Series Ideas Add Up to Interesting Mathematics

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Introduction

Infinite series prove important in numerous mathematics applications and, consequently, appear in a number of mathematics courses including analysis, calculus, and differential equations. High school analysis often addresses geometric series; calculus courses often devote a great deal of coverage time to the study of convergence of infinite series. Unfortunately, most students dislike the subject immensely. To stimulate student interest in infinite series, I propose the introduction of geometric examples in which some seeming paradoxes arise.

I illustrate with three particular examples – two fractals and an integration-less variation on Gabriel's Horn. Before delving into these examples, let's recall some special types of series.

Some Special Infinite Series

The examples I wish to consider involve three categories of series: geometric series, harmonic series, and p-series. A geometric series consists of a sum of the form

$$1 + r + r^2 + r^3 + r^4 + \cdots .$$

One typically evaluates a geometric series using the simplification

$$1 + r + r^2 + r^3 + r^4 + \cdots + r^n = \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} .$$

This shortcut requires that $r \neq 1$. In the particular case $r = 1$, the above series sums to $n + 1$.

We evaluate the *infinite* geometric series $\sum_{i=0}^{\infty} r^i$ by considering the limit of the finite series as *n* tends to infinity. Noting that r^{n+1} tends to infinity if $|r|>1$, and considering as special cases $r = 1$ and $r = -1$, we find that the infinite series converges to a finite value only when $|r| < 1$, and that the series then sums to

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

(note that if $r \geq 1$, the series diverges to ∞). For instance, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \frac{1}{1-1/2} = 2$.

A second type of series we will need is the harmonic series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{i=1}^{\infty} \frac{1}{i}.$$

This series is known to diverge to infinity.

The third and final category of series we need is the p-series; this is a series of the form

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p ,$$

where *p* is a positive real number. Traditionally, one uses the Integral Test for Series Convergence to see that the p-series converges to a finite value for $p > 1$. Unfortunately, the integral test does not tell us the value to which the series converges. However, Leonard Euler **[6]** considered one special case that will be needed shortly:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{p^2}{6}.$$

Example 1 –Sierpenski's Triangle

We begin our fractal examples by considering Sierpenski's triangle. We first consider this figure through an activity referred to as the "chaos game" **[3]** by fractal expert Michael Barnsley.

The chaos game is a fun, stochastic method for constructing an image of the Sierpenski triangle. But, for further analysis of the figure, we'll need to look at the deterministic construction suggested in Figure 1. The deterministic construction begins with a filled equilateral triangle and proceeds in an iterative manner. At each iteration, we remove an equilateral triangle formed by connecting the midpoints of the sides of each filled triangle.

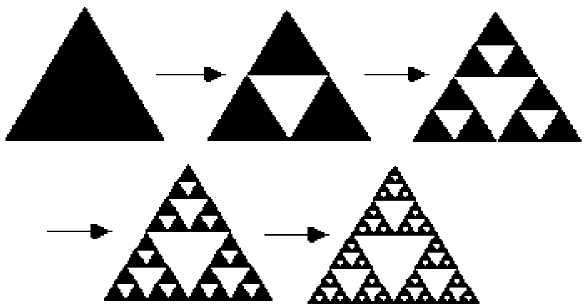


Figure 1: Constructing the Sierpenski Triangle

The Sierpenski triangle possesses some curious properties with regard to its area and perimeter. We determine the area and perimeter of the figure by examining the stages of the iteration scheme as indicated in Table 1. While its area is zero, we see that the perimeter is infinite using a geometric series with $r = 3/2$ ($r > 1$).

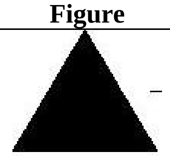
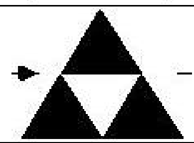
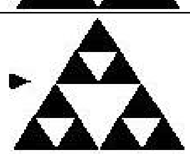
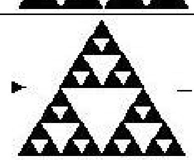
Figure	**Area**	**Perimeter**
	$A_0 = \frac{\sqrt{3}}{4}$	$P_0 = 3$
	$A_1 = \frac{3}{4} A_0$	$P_1 = 3 + 3(1/2)$ $= 3 + 3/2$
	$A_2 = (3/4)^2 A_0$	$P_2 = 3 + 3/2 + 3(3)\,¼$ $= 3 + 3/2 + (3/2)^2$
	$A_3 = (3/4)^3 A_0$	$P_3 = 3 + 3/2 + (3/2)^2 + 9(3)(1/8)$ $= 3 + 3/2 + (3/2)^2 + (3/2)^3$
Stage *n*	$A_n = (3/4)^n A_0$	$P_n = 3 + 3/2 + \cdots + (3/2)^n$
Sierpenski triangle	0	∞

Table 1: Determining Area and Perimeter of Sierpenski Triangle

Example 2 – The Koch Snowflake

Lest we get the impression that this area-perimeter property is a fluke, let's consider another fractal figure, the Koch snowflake **[2]**. The snowflake is constructed via an iterative process as suggested in Figure 2. As in our consideration of the Sierpenski triangle, we analyze the area and perimeter of the figure in stages, as outlined in Table 2. With the Koch snowflake we find yet another figure having finite area, but infinite perimeter.

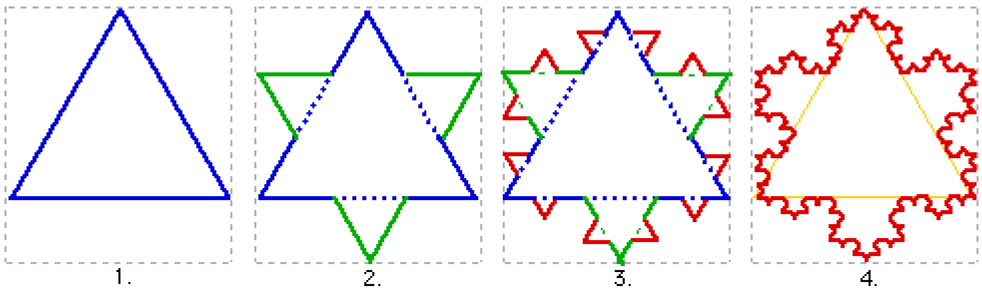


Figure 2: Koch Snowflake Construction

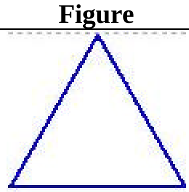
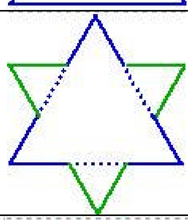
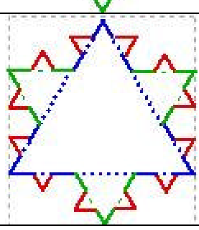
Figure	Num. Faces	Area	Perimeter
	3	$A_0 = \dfrac{\sqrt{3}}{4}$	$P_0 = 3$
	4(3)	$A_1 = A_0 + 3A_0(1/3)^2 = A_0[1+1/3]$	$P_1 = 4(3)(1/3) = 3(4/3)$
	$4(4)(3) = 4^2(3)$	$A_2 = A_1 + 4(3)A_0(1/3)^4 = A_0\left[1+(1/3)+4(1/3)^3\right]$	$P_2 = 4^2(3)(1/3)^2 = 3(4/3)^2$
stage *n*	$4^n(3)$	$A_0\left[1+1/3+4(1/3)^3+\cdots+4^{n-1}(1/3)^{2n-1}\right]$	$P_n = 3(4/3)^n$
snowflake	∞	$\dfrac{2\sqrt{3}}{5}$	∞

Table 2: Area and Perimeter of the Koch Snowflake

Example 3 – Gabriel's Wedding Cake

Interesting examples are not just found among fractals. A famous example involves Gabriel's Horn, a three dimensional solid having finite volume and infinite surface area. Since analyzing Gabriel's Horn requires integration, we instead consider a figure referred to as Gabriel's wedding cake by its creator, Julian Fleron **[4]**.

We construct Gabriel's wedding cake by revolving the graph of a step function about the x-axis:

$$f(x) = \begin{cases} 1, & \text{if } 1 \le x < 2 \\ 1/2, & \text{if } 2 \le \text{x} < 3 \\ & \vdots \\ 1/n, & \text{if } n \le x < n+1 \\ & \vdots \end{cases} .$$

Each layer of the cake is a cylinder. We determine the volume of the wedding cake by summing the volumes of the cylinders:

$$V = \sum_{n=1}^{\infty} p \, r_n{}^2 = \sum_{n=1}^{\infty} p \left(\frac{1}{n}\right)^2 .$$

Using Euler's p-series formula for p = 2, we then see that the volume is $p^3/6$.

We determine the surface area by considering the tops and sides of the layers separately. The top of each layer forms an annulus. The area of the n^{th} annulus is given by

$$p\left(\frac{1}{n}\right)^2 - p\left(\frac{1}{n+1}\right)^2 ,$$

so that the areas of the tops summed together yield a telescoping series

$$\text{Area of tops} = \sum_{n=1}^{\infty} \left[p\left(\frac{1}{\text{n}}\right)^2 - p\left(\frac{1}{n+1}\right)^2 \right] = p .$$

The lateral area of the n^{th} layers is given as

$$\text{(circumference)(height)} = 2pr(1) = \frac{2p}{n}.$$

Adding up the lateral areas of the layers, we get an infinite series

$$\text{Area of layer sides} = \sum_{n=1}^{\infty} \frac{2p}{n} = 2p \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

since this is a multiple of the harmonic series.

Thus we see that one can make enough dough to bake Gabriel's cake, but cannot make enough frosting to cover it.

References

1. Michael Barnsley, *Fractals Everywhere*, Academic Press, Boston (1988).
2. Bellevue Community College, "The Snowflake Curve" web site at http://scidiv.bcc.ctc.edu/Math/Snowflake.html.
3. Robert Devaney, "Chaos in the Classroom" web site at http://math.bu.edu/DYSYS/chaos-game/chaos-game.html.
4. Julian Fleron, "Gabriel's Wedding Cake," *The College Mathematics Journal* (30), 1: 35-38 (1999).
5. R. Larson, R. Hostetler, and B. Edwards, "Exercise 74: Sphereflake", *Calculus with Analytic Geometry, Sixth Edition*, pp. 545 and 566, Houghton Mifflin Company, Boston (1998).
6. R. Young, "Summing the Reciprocals of the Squares", appearing in *Excursions in Calculus*, pp. 338-56, Mathematical Association of America (1992).