Predicting Limits for Boltzmann-type ODEs

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July 12, 1996

2nd World Congress of Nonlinear Analysts

Introduction

$$y' + y = f(y)$$

$$\text{y} \in R^n,\ y(0) \in C$$

$C \subset R^n$ closed, convex

$f : C \to C$ purely quadratic

Conditions on $f \Rightarrow$
$\exists w \in C,\ \lim_{t \to \infty} j(t, y_0) = w$

$$y' + y = f(y) + Ay$$

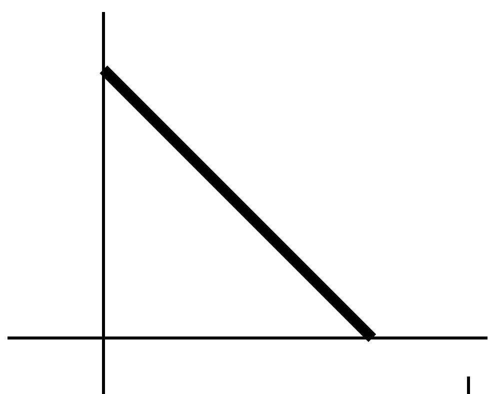
A $n \times n$ real matrix

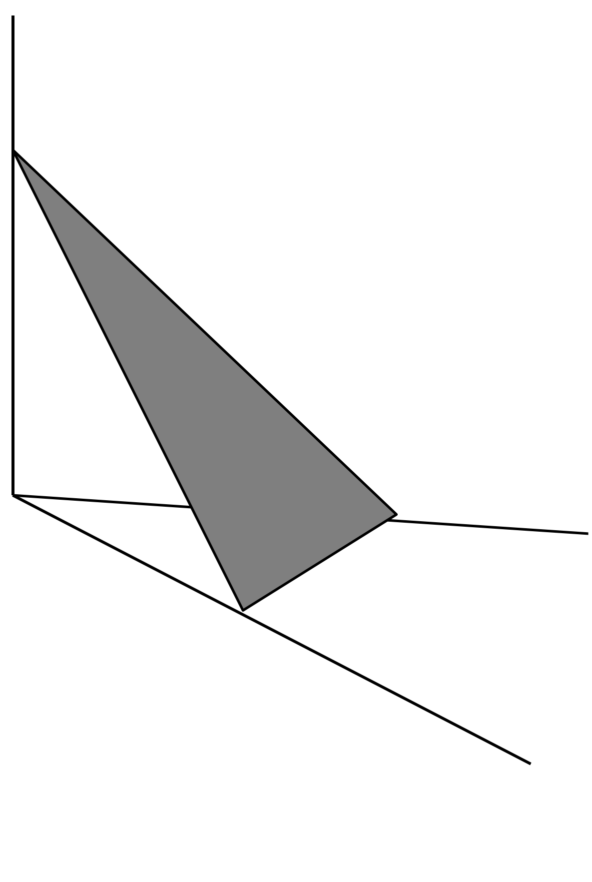
Conditions on $A \Leftrightarrow$
$\exists w \in C,\ \lim_{t \to \infty} j(t, f, A, y_0) = w\ \forall f, y_0$

The Setting

$$C = U \cap R_+^n$$

$$U = \left\{ x \in R^n : \sum_i x_i = 1 \right\}$$





The Structure of *f*

1. <u>purely quadratic:</u> $f(y) = Q(y, y)$,

with

$$Q(x,y) = \left(\left\langle x, Q_1 y \right\rangle, \left\langle x, Q_2 y \right\rangle, \ldots, \left\langle x, Q_n y \right\rangle \right)$$

2. $f : C \to C$

(a). $Q_i(p,q) \geq 0$ for each i, p, q

(b). $\sum_{i=1}^{n} Q_i(p,q) = 1$ for each p, q

Example, Structure of *f*

$$Q_1 = \begin{pmatrix} 1 & 2/3 & 1/3 \\ 2/3 & 0 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 1/6 & 1/3 \\ 1/6 & 1 & 1/2 \\ 1/3 & 1/2 & 0 \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} 0 & 1/6 & 1/3 \\ 1/6 & 0 & 1/2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$$

$$f(y) = \begin{pmatrix} y_1^2 + 4y_1y_2/3 + 2y_1y_3/3 \\ y_2^2 + y_1y_2/3 + 2y_1y_3/3 + y_2y_3 \\ y_3^2 + y_1y_2/3 + 2y_1y_3/3 + y_2y_3 \end{pmatrix}$$

Background from Tjon Wu

$u(t,x) = \%$ with energy x at time t

$$\frac{\P u}{\P t}(t,x) = \int_0^{\infty}\int_0^{\infty} P(y,z|x)\ u(t,y)\ u(t,z)\ dy\ dz$$

$$-u(t,x)\int_0^{\infty} u(t,y)\ dy$$

$$= \int_0^{\infty}\int_0^{\infty} P(y,z|x)\ u(t,y)\ u(t,z)\ dy\ dz$$

$$-u(t,x)$$

$$X = L^1(0,\infty)$$

$$C = \left\{ u \in L^1 : u \geq 0 \text{ and } \int_0^{\infty} u(s)ds = 1 \right\}$$

$$Q(u,v) = \iint P(y,z|x)\ u(y)\ v(z)\ dy\ dz$$

Logarithmic Norm:

$$m(T) := \lim_{h \downarrow 0} \frac{Lip(I + hT) - 1}{h}$$

Properties:

$$m(f+g) \leq m(g) + m(g)$$

$$|m(f)| \leq Lip(f)$$

$$m(f + aI) = m(f) + \mathrm{Re}(a)$$

Theorem [Dahlquist]:

If A is a $n \times n$ real matrix then, with respect to the l_1 norm,

$$m(A) = \max_k \left\{ a_{kk} + \sum_{i \neq k} |a_{ik}| \right\}.$$

Theorem [Herod]:

$$m(Q) = \max_{p,q,r} \Big\{ Q_q(p,q) - Q_q(p,r) + Q_r(p,r) - Q_r(p,q) + \sum_{i \neq p,q} |Q_i(p,q) - Q_i(p,r)| \Big\}$$

Example 1: $A = \begin{pmatrix} -4 & 2 & -1 \\ -1 & -5 & -1 \\ 0 & 1 & -3 \end{pmatrix}$

$m(A) = -1$

Example 2:

$$Q_1 = \begin{pmatrix} 1 & 2/3 & 1/3 \\ 2/3 & 0 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 1/6 & 1/3 \\ 1/6 & 1 & 1/2 \\ 1/3 & 1/2 & 0 \end{pmatrix}$$

$$Q_3 = \begin{pmatrix} 0 & 1/6 & 1/3 \\ 1/6 & 0 & 1/2 \\ 1/3 & 1/2 & 1 \end{pmatrix}$$

$m(Q) = 5/3$

Theorem [Martin]:

If u and v are solutions to $y' = g(y)$,

$$|u(t) - v(t)| \le e^{m(g)t} |u(0) - v(0)| \ \ \forall t \ge 0.$$

$$y' + y = f(y) \ \Rightarrow \ y' = f(y) - y$$

Theorem:

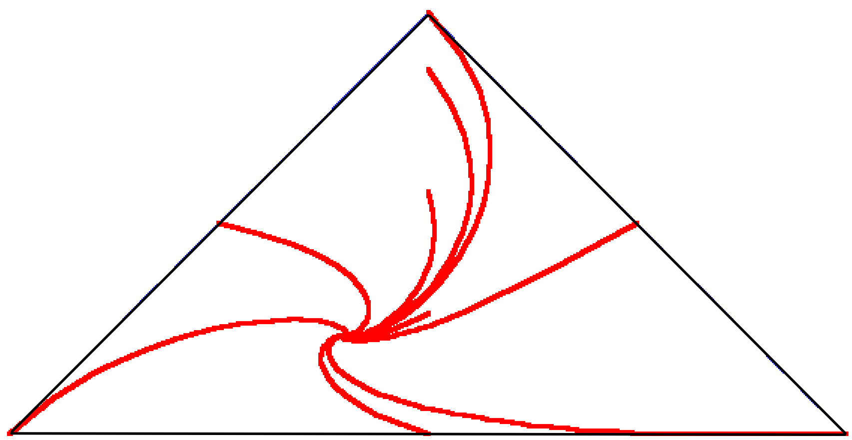
$m(f) \le 1 \Rightarrow \exists\, w \in C$ such that

$\lim_{t \to \infty} y(t) = w$ for any $y(0) \in C$.

Example:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad m(Q) = 0$$



$$y' + y = f(y) + Ay$$

Theorem:

$$y(t) \in U \ \ \forall t \geq 0, \ \forall f \quad \Leftrightarrow \sum_j A_{ij} = 0 \ (\forall i).$$

Theorem:

A as above. $y(t) \in R_+^n \ \forall t \geq 0$,

$\forall f: C \to C$, each $y(0) \in C \Leftrightarrow$

$A_{ij} \geq 0 \ (i \neq j)$.

Def: A admissable perturbation:

$$A_{ij} \geq 0 \ (i \neq j),$$

$$\sum_j A_{ij} = 0 \ (\forall i).$$

A admissable $\Rightarrow m(A) = 0.$

Corollary:

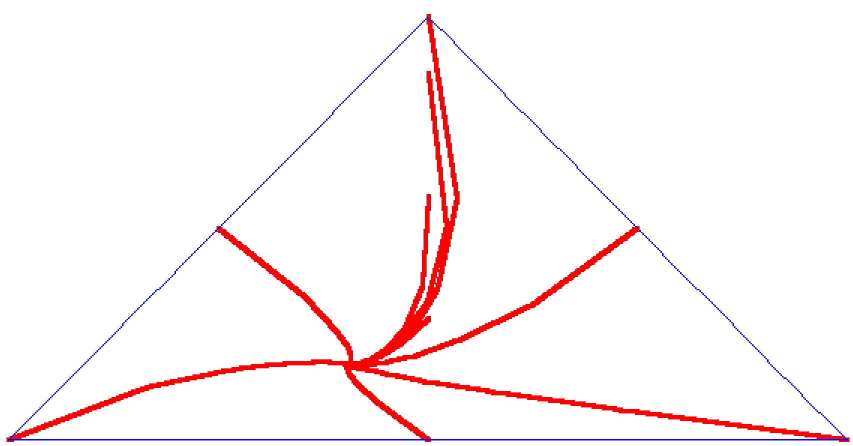
A admissable, $m(f) \leq 1 \ \Rightarrow$

$\exists w \in C$ such that $\lim_{t \to \infty} y(t) = w \quad (\forall y_0 \in C).$

Example:

Q as in previous example

$$A = \begin{pmatrix} -2 & 2 & 2 \\ 1 & -3 & 3 \\ 1 & 1 & -5 \end{pmatrix}$$



<u>The Control Question:</u>

Given f, can we select an admissable perturbation which drives all solutions to a specified point w?

Unreachable Points:

$$Q_1 = Q_2 = \begin{pmatrix} 1/2 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are unreachable. For $(1,0,0)$ to be reachable, must have

$$Q_1 = \begin{pmatrix} 1 & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{pmatrix},\ A = \begin{pmatrix} 0 & \# & \# \\ 0 & \# & \# \\ 0 & \# & \# \end{pmatrix}$$

References:

1). Dahlquist, “Stability and error bounds in the numerical integration of ordinary differential equations”, **Trans. of the Royal Inst. of Tech., Stockholm, Sweden,** 1959, Number 130.

2). J.V. Herod, “Asymptotic properties for the differential equation Y’+Y=F(Y)”, Georgia Tech preprint, Math code: 070792-011, 1992.

3). T. Howard, “Predicting asymptotic behavior for Boltzmann-type ordinary differential equations”, submitted for publication.

4). R.H. Martin, Jr., “Existence and bounds of solutions to ordinary differential equations in a Banach space”, PhD thesis, Georgia Tech, 1970.