Something for Everyone: Approximation Theme Provides Carry-over

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Innovations in Undergraduate Mathematics
May 7, 1996

INTRO -- the error term

APPROXIMATION METHODS

Pre-calculus: point fitting

Differential Calc: match derivatives

Integral Calc: match moments

Multivariate: least squares fit

WHICH IS BEST?

$L_2[-a,a]$ criterion

Moment Equations

$$\cos(x)\,dx = (ax^2 + bx + c)\,dx$$

$$x\cos(x)\,dx = x\,(ax^2 + bx + c)\,dx$$

$$x^2\cos(x)\,dx = x^2\,(ax^2 + bx + c)\,dx$$

Moment Approximation

$$2 = A\,\pi^3/12 + C\,\pi$$

$$0 = B\,\pi^3/12$$

$$\pi^2/4 - 4 = A\,\pi^5/80 + C\,\pi^3/12$$

$$q(x) = x^2\,(60\pi^2-12)/\pi^5 - 3(\pi^2-20)/\pi^3$$

Least Squares Method

$$F(a,b,c) = \Sigma_i \, (ax_i^2 + bx_i + c - y_i)^2$$

$$\partial F/\partial a = \Sigma_i \, 2x_i^2 (ax_i^2 + bx_i + c - y_i)$$

$$\partial F/\partial b = \Sigma_i \, 2x_i (ax_i^2 + bx_i + c - y_i)$$

$$\partial F/\partial c = \Sigma_i \, 2 (ax_i^2 + bx_i + c - y_i)$$

Least Squares Solution

$$0 = a\, 2\pi^3/3 + c\, 2\pi$$

$$0 = b\, 2\pi^3/3$$

$$-4\pi = a\, 2\pi^5/5 + c\, 2\pi^3/3$$

$$q(x) = -\, 45\, x^2 / (2\, \pi^4) + 15 / (2\pi^2)$$

Theorem

Let f be an even function with bounded variation on [-a, a]. Then the best quadratic approximation for f on [-a, a] in the L_2 sense is the one having the same 0^{th}, 1^{st}, and 2^{nd} moments as f.

A Maple-assisted proof

```
> eq1 := 2*int( f(t), t=0..a )=A*(2*a^3/3) + C*(2*a);

> eq2 := 0 = B*(2*a^3/3);

> eq3 := 2*int(t^2*f(t), t=0..a ) = A*(2*a^5/5) +
          C*(2*a^3/3);

> solve( { eq1, eq2, eq3 }, { A, B, C } );

> assign( " );

> qmom := x -> A*x^2 + B*x + C;
```

```
> F := (a,b,c) -> int( ( a*x^2 + b*x + c - f(x) )^2,
                        x=-a..a );
> eq1 := diff( F(a,b,c), a ) = 0;
> eq2 := diff( F(a,b,c), b ) = 0;
> eq3 := diff( F(a,b,c), c ) = 0;
> solve( { eq1, eq2, eq3 }, { a, b, c } );
```

```
> with( linalg );

> H := matrix( 3,3, [ seq(seq( diff( F( u1, u2, u3),
        u.`i`, u.`j`), i=1..3 ), j=1..3 );
```

$$\begin{matrix} 4a^5/5 & 0 & 4a^3/3 \\ 0 & 4a^3/3 & 0 \\ 4a^3/3 & 0 & 4a \end{matrix}$$

```
> det( H );
```

$$256\, a^9 / 135$$

```
> eigenvalues( H );
```

$$4a^3/3 \,,$$

$$2a^5/5 + 2a + 2a/15\ \mathrm{sqrt}(\ 9a^8 + 10a^4 + 225\ ) \,,$$

$$2a\ (\ a^4/5 + 1 - \mathrm{sqrt}(\ 9a^8 + 10a^4 + 225\ ) / 15\ )$$