Test 4 Solutions, Differential Equationss,
Dr. Howard, Spring 1999

1. Transform the following differential equation into an equivalent first-order system:

$$x'' - 2x' + 3x = 0.$$

Answer: Set $y = x'$. The second order differential equation is equivalent to the following first-order system:

$$x' = y$$
$$y' = -3x + 2y.$$

2. Given the following system of differential equations and initial values, approximate the values of $x(0.1)$ and $y(0.1)$ using Euler's method with a step size $h = 0.1$:

$$x' = x + 2y, \ \ x(0) = 0$$

$$y' = 2x + y, \ \ y(0) = 2.$$

Answer:

$$x_0 = 0$$
$$y_0 = 2$$
$$x_1 = x_0 + hf_1(x_0, y_0) = 0 + 0.1(0 + 2(2)) = 0.4$$
$$y_1 = y_0 + hf_2(x_0, y_0) = 2 + 0.1(2(0) + 2) = 2.2.$$

3. Write the following system of differential equations as a system in the form $\mathbf{x}' = P(t)\mathbf{x} + f(t)$:

$$x' = 2x + 4y + 3e^t$$

$$y' = 5x - y - t^2.$$

Answer:

Let $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $P(t) = \begin{pmatrix} 2 & 4 \\ 5 & -1 \end{pmatrix}$, and $f(t) = \begin{pmatrix} 3e^t \\ -t^2 \end{pmatrix}$.

4. Use the eigenvalue method to determine the general solution of the system of differential equations:

$$x_1' = 2x_1 + 3x_2$$

$$x_2' = 4x_2$$

Answer:

Write as the system $Y' = AY$, with $Y(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$. The characteristic polynomial is $p(\lambda) = (2 - \lambda)(4 - \lambda)$, so the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Therefore, the general solution to the system of differential equations is $Y(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

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5. Given that the matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$, with corresponding eigenvectors $\mathbf{v}_1 = (1\ {-1})^T$ and $v_2 = (3\ \ 2)^T$, find the solution of the initial value problem

$$Y' = AY, \quad Y(0) = \begin{pmatrix} 0 \\ -5 \end{pmatrix}.$$

Answer: The solution will be of the form $Y(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where c_1 and c_2 are chosen so that $\begin{pmatrix} 0 \\ -5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, that is $\begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$. The solution is $c_1 = 3$ and $c_2 = -1$, so the solution to the initial value problem is

$$Y(t) = 3e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

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6. The matrix $A = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}$ has one eigenvalue, $\lambda = 4$ with defect = 1, and an eigenvector $\mathbf{v} = (1\ \ {-1})^T$. Find the general solution of the system of differential equations $Y' = AY$.

Answer:

Finding the generalized eigenvector:

$(A - \lambda I)v_2 = v_1$: $\begin{pmatrix} -3 & -3 \\ 3 & 3 \end{pmatrix} v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ has solutions of the form $v_2 = \begin{pmatrix} -\frac{1}{3} - b \\ b \end{pmatrix}$. Taking $b = 0$, we then have that the general solution of the system can be written as $Y(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t} \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix} \right\}$.