Differential Equations Test 2 Solutions
Dr. Howard, Spring 1999

1. Use Euler's method with a step size of $h = 0.5$ to estimate $y(2)$ given the initial value problem

$$\frac{dy}{dx} = 2x + y, \quad y(1) = 1.5. \qquad \textbf{Answer}: \; y(2) \approx 6.375.$$

2. Consider the differential equation $\frac{dy}{dt} = y^2 - 6y - 16$ in responding to each part below.

a. Sketch the associated phase line.

Anwer: Unable to typeset phase line. The phase line should indicate two equilibria: $y = -2$ and $y = 8$.

b. Classify each equilibrium as unstable or asymptotically stable.

Answer: $y = -2$ is asymptotically stable and $y = 8$ is unstable.

c. Describe the solution corresponding to $y(0) = 3$ by indicating two things:

i. Describe whether $y(t)$ is increasing or decreasing. **Answer**: decreasing.

ii. Determine whether or not $\lim_{t\to\infty} y(t)$ exists. Find the limit if it does exist.

Answer: $\lim_{t\to\infty} y(t) = -2$.

3. Consider the differential equation $\frac{dy}{dt} = y^2 - k + 1$, where k is a parameter.

a. Sketch the phase lines associated with $k = 0$ and $k = 5$.

Answer: For $k = 0$ no equilibria; for $k = 5$ there are equilibria at $y = \pm 2$.

b. Determine the value of k at which a bifurcation takes place. **Answer**: $k = 1$.

4. Suppose that a motorboat is moving at a speed of 30 ft / sec when its motor suddenly quits, and that 10 sec later the boat has slowed to a speed of 15 ft / sec. Assume that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?

Answer: $\lim_{t\to\infty} x(t) = \frac{300}{\ln 2} \approx 432.8$ feet.

5. Suppose a species of fish in a particular lake has a population that is modeled by the logistic population model with growth rate r, carrying capacity K, and time t measured in years. For each part below, write down a differential equation to model the described situation by adjusting the logistic growth model appropriately.

a. Scenario 1: A constant amount of 100 fish is harvested each year.

Answer: $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - 100.$

b. Scenario 2: The number of fish harvested each year is proportional to the square root of the number of fish in the lake.

Answer: $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \delta\sqrt{P}.$