The Geometry of Linear Systems

Suppose that A is a 2×2 real matrix with **nonzero** eigenvalues. In class, we'll discuss the structure of solutions of the linear system of differential equations

$$Y' = AY$$

(that is, what are the possible shapes of solution curves in the phase plane). Exactly what the geometry will be like depends on the eigenvalues and eigenvectors of A. The possible cases are outlined below. Assume the eigenvalues are λ_1 and λ_2, with corresponding eigenvectors v_1 and v_2. In all cases, draw representative solution curves, then use arrows to indicate the direction of flow along the curve as t increasing. A point at the origin indicates that it is always a stationary solution (equilibrium point) of the system.

1. A has two distinct, real eigenvalues.

   Then if the initial condition lies on one of the eigenspaces, the solution curve is a straight line through that point and the origin. If the initial condition does not lie on one of the eigenspaces, see the cases below.

   a. Both eigenvalues have the same sign, with $|\lambda_1| > |\lambda_2|$.

      In this case, the origin is called an *improper node*. If the initial vector does not lie on an eigenspace, then the solution curve will be tangent to the v_2-eigenspace.

   b. $\lambda_1 < 0 < \lambda_2$.

      Then the origin is called a *saddle point*. If the initial vector lies off of the eigenspaces, the solution will initially head toward the origin, ”trying” to follow the direction of the v_1-eigenspace, but will turn and veer away and approach the v_2-eigenspace.

2. A has only one distinct (real) eigenvalue λ (so $\lambda_1 = \lambda_2 = \lambda$).

   Solutions that start on an eigenspace remain on the eigenspace for all time. If the initial point does not lie in an eigenspace, see below.

   a. λ has zero defect (two linearly independent eigenvectors).

      All solutions are straight-line solutions. The origin is a *proper node*.

   b. λ has defect equal to one.

      The origin is called an *improper node*. There is only one eigenspace, determined by v_1. Solution curves corresponding to initial conditions lying off the eigenspace are tangent to the eigenspace at the origin; test a few initial points to see the general orientation of these trajectories.

3. A has complex conjugate eigenvalues, $\lambda = \alpha \pm i\beta$ (with $\beta \neq 0$).

   a. $\alpha \neq 0$.

      In this case, all solution curves (except the one starting at the origin) spiral around the origin, which is called a *spiral point*. Determine the orientation of the spiral by looking at the sign of α and testing a couple of initial points.

   b. $\alpha = 0$.

      In this case, all solution curves (except the one starting at the origin) orbit the origin in an elliptical pattern. The origin is called a *center*. Determine the orientation of the ellipse by testing a couple of initial points. The major and minor axes of the ellipse can be determined by examining the eigenvectors.