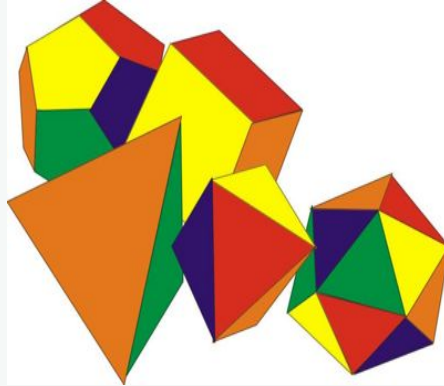


Platonic solids “in \mathbb{Z}^3 ”



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joint work with A. Markov

An example of a regular tetrahedron and a cube in \mathbb{Z}^3 :

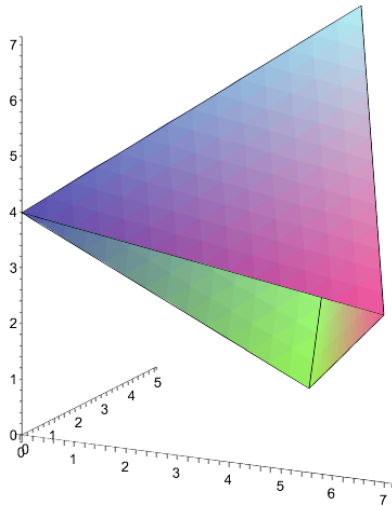


Figure 1a, $\{(0,0,4),(7,0,3),(3,5,0),(4,5,7)\}$

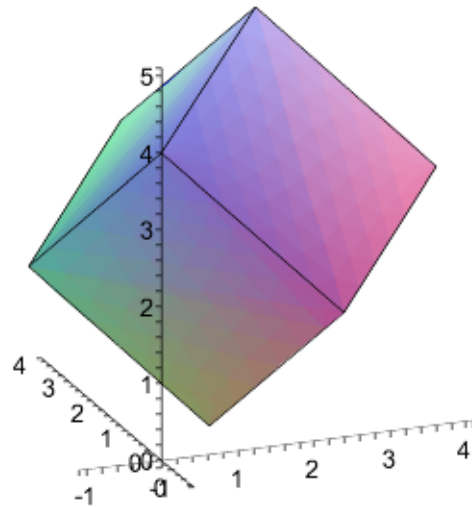


Figure 1b, $(1,4,3),(3,3,1),(1,1,0),(-1,2,2),(2,2,5),(4,1,3),(2,-1,2),(0,0,4)$

Euclidean Distance: $d(A, B) := \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}$, where $A = (a_x, a_y, a_z)$ and $B = (b_x, b_y, b_z)$

An example of a regular octahedron in \mathbb{Z}^3 :

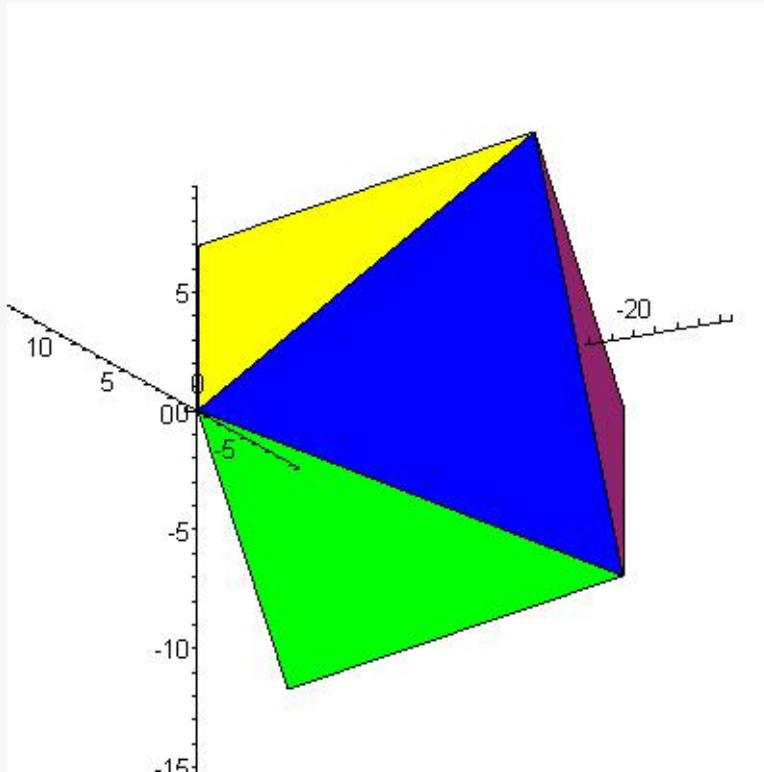


Figure 2, $\{(0,0,0), (1,-16,9), (-8,-15,-7), (16,-9,1), (7,-8,-15), (8,-24,-6)\}$

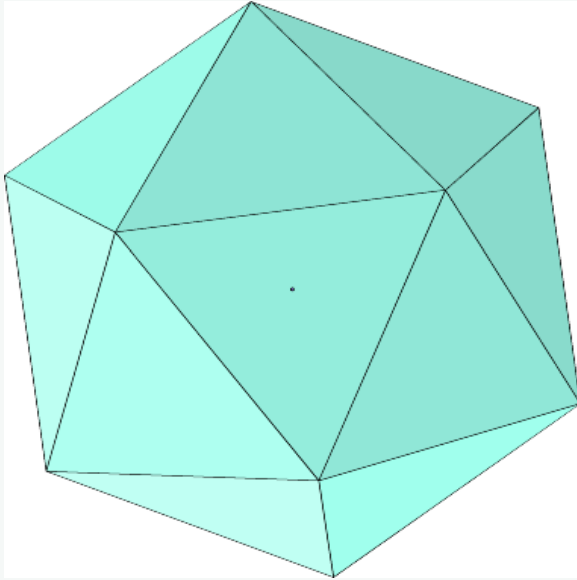


Figure 3 (a): Regular Icosahedron

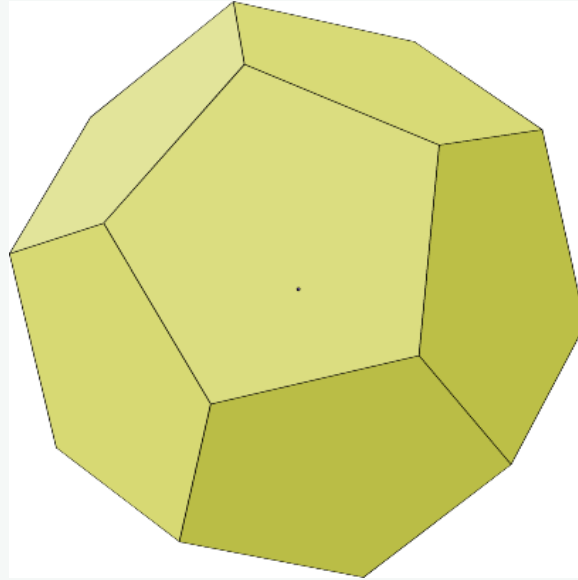
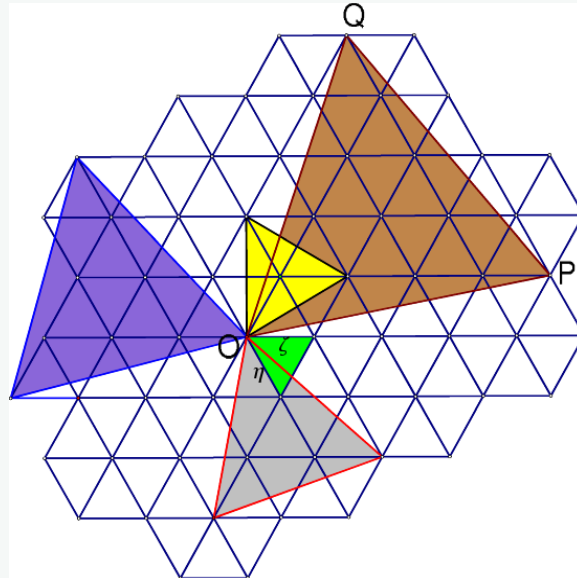


Figure 3 (b): Regular Dodecahedron

Theorem A: *There is no regular icosahedron or regular dodecahedron in \mathbb{Z}^3 .*

The lattice $\mathcal{P}_{a,b,c}$

$$\mathcal{P}_{a,b,c} := \{(\alpha, \beta, \gamma) \in \mathbb{Z}^3 \mid a\alpha + b\beta + c\gamma = 0, \quad a^2 + b^2 + c^2 = 3d^2, \quad a, b, c, d \in \mathbb{Z}\}. \quad (1)$$



Fact: *The cosine of the dihedral angle between two adjacent faces of a regular icosahedron is $-\frac{\sqrt{5}}{3}$ (for regular tetrahedron is $\frac{1}{3}$, and for regular octahedron is $\frac{-1}{3}$).*

Parametrization (JIS '07 and Integers '08):

Let a, b, c, d be odd integers such that $a^2 + b^2 + c^2 = 3d^2$ and $\gcd(a, b, c) = 1$. Then for every $m, n \in \mathbb{Z}$ (not both zero) the triangle OPQ (O being the origin), determined by

$$\overrightarrow{OP} = m\vec{\zeta} - n\vec{\eta}, \quad \overrightarrow{OQ} = m(\vec{\zeta} - \vec{\eta}) - n\vec{\zeta}, \quad \text{with } \vec{\zeta} = (\zeta_1, \zeta_2, \zeta_3), \vec{\eta} = (\eta_1, \eta_2, \eta_3), \quad (2)$$

$$\left\{ \begin{array}{l} \zeta_1 = -\frac{rac + dbs}{q}, \\ \zeta_2 = \frac{das - bcr}{q}, \\ \zeta_3 = r, \end{array} \right. , \quad \left\{ \begin{array}{l} \eta_1 = -\frac{db(s - 3r) + ac(r + s)}{2q}, \\ \eta_2 = \frac{da(s - 3r) - bc(r + s)}{2q}, \\ \eta_3 = \frac{r + s}{2}, \end{array} \right. \quad (3)$$

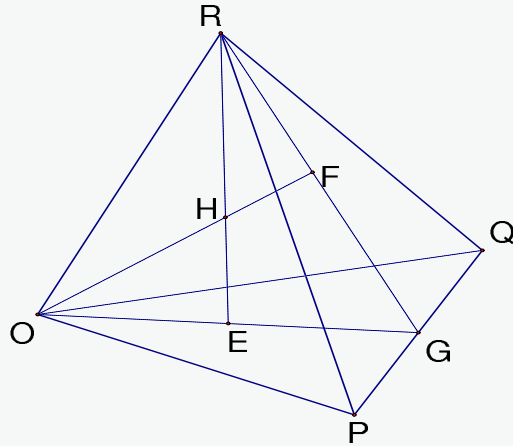
where $q = a^2 + b^2$ and (r, s) is a suitable solution of $2q = s^2 + 3r^2$ that makes all the numbers in (3) integers, forms an equilateral triangle in \mathbb{Z}^3 contained in the lattice $\mathcal{P}_{a,b,c}$ and having sides-lengths equal to $d\sqrt{2(m^2 - mn + n^2)}$.

Conversely, there exists a choice of the integers r and s such that given an arbitrary equilateral triangle in \mathbb{R}^3 whose vertices, one at the origin and the other two in the lattice $\mathcal{P}_{a,b,c}$, then there exist integers m and n such that the two vertices not at the origin are given by (2) and (3).

Example: $d = 41$, $(1^2 + 1^2 + 71^2 = 3d^2)$, $m = 55$, $n = 16$, $49^2 = m^2 - mn + n^2$

$$\overrightarrow{OP} = (2840, -71, -39), \overrightarrow{OQ} = (1359, -2495, 16), \quad \text{side lengths } 2009\sqrt{2}$$

Regular Tetrahedra



$$\Omega(k) := \{(m, n) \in \mathbb{Z}^2 : m^2 - mn + n^2 = k^2\}$$

$$R = \left(\frac{(2\zeta_1 - \eta_1)m \pm 2ak}{3}, \frac{(2\zeta_2 - \eta_2)m - (\zeta_2 + \eta_2)n \pm 2bk}{3}, \frac{(2\zeta_3 - \eta_3)m - (\zeta_3 + \eta_3)n \pm 2ck}{3} \right), (m, n) \in \Omega(k). \quad (4)$$

Theorem (JNT 2009)

Every regular tetrahedron in \mathbb{Z}^3 having one of its vertices the origin and side lengths $\lambda\sqrt{2}$, can be obtained by taking as one of its faces an equilateral triangle described by the previous Parametrization in which with a, b, c and d odd integers satisfying $a^2 + b^2 + c^2 = 3d^2$ with d a divisor of λ , and then completing it with the fourth vertex as in (4) for some $(m, n) \in \Omega(\frac{\lambda}{d})$.

Conversely, if we let a, b, c and d be a primitive solution of $a^2 + b^2 + c^2 = 3d^2$, let $k \in \mathbb{N}$ and $(m, n) \in \Omega(k)$, then the coordinates of the point R in (4) are

- (i) all integers, if $k \equiv 0 \pmod{3}$ regardless of the choice of signs or
- (ii) integers, precisely for only one choice of the signs if $k \not\equiv 0 \pmod{3}$.

Example: $R_1 := [\frac{4297}{3}, \frac{-2468}{3}, \frac{6935}{3}]$, $R_2 := [1367, -888, -2327]$

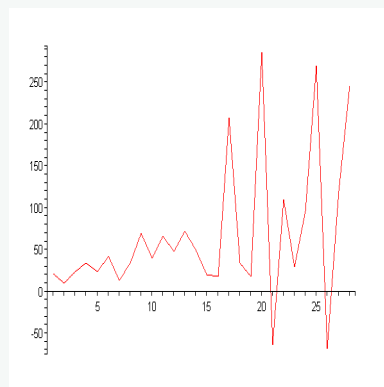
The number of regular tetrahedra whose coordinates of its vertices are in the set $\{0, 1, \dots, n\}$ is the sequence [2A103158](#) Link: [A103158](#) (2005)

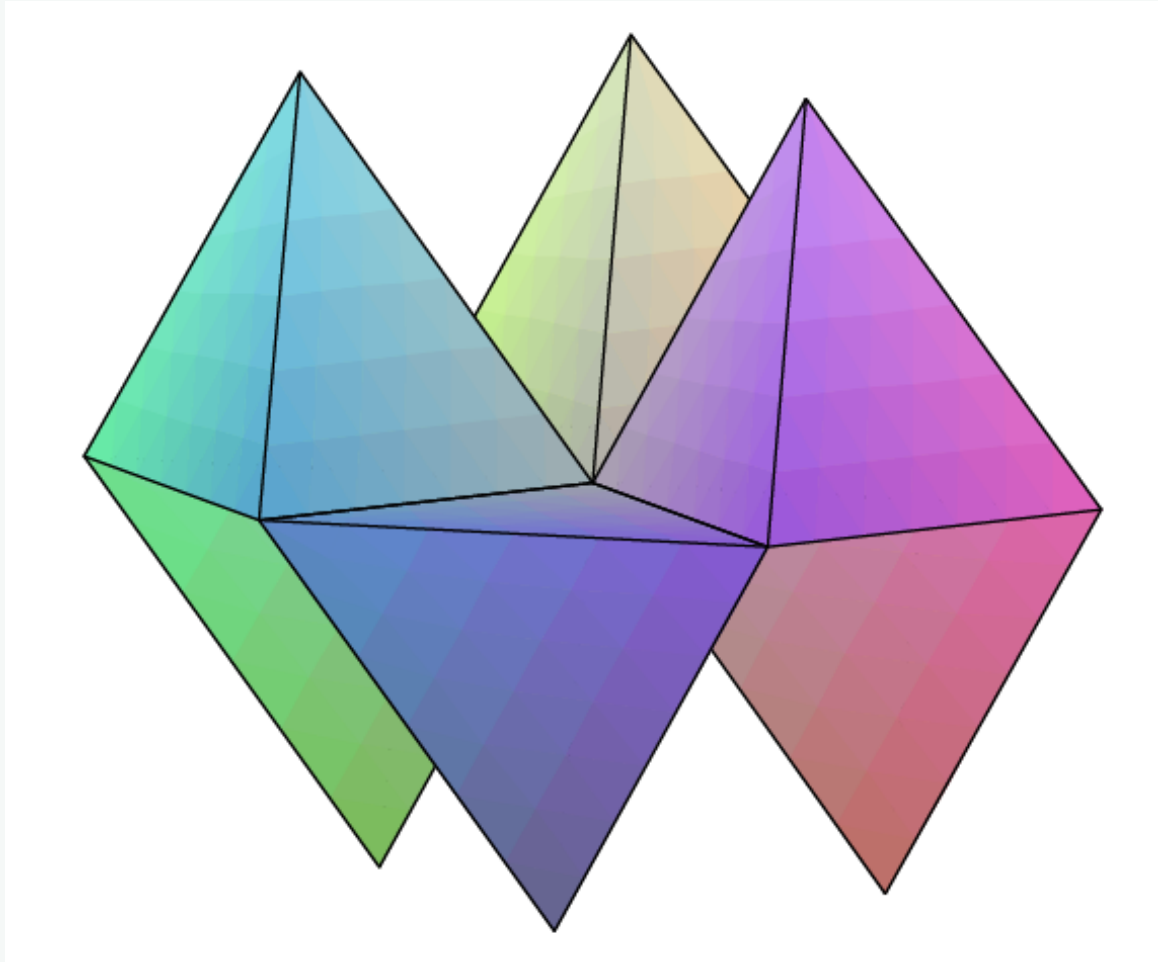
n	1	2	3	4	5	6	7	8	9	10	11
A103158	1	9	36	104	257	549	1058	1896	3199	5154	7926

n	12	13	14	15	16	17	18
A103158	11768	16967	23859	32846	44378	58977	77215

n	19	20	21	22	23	24	25
A103158	99684	126994	159963	199443	246304	301702	366729

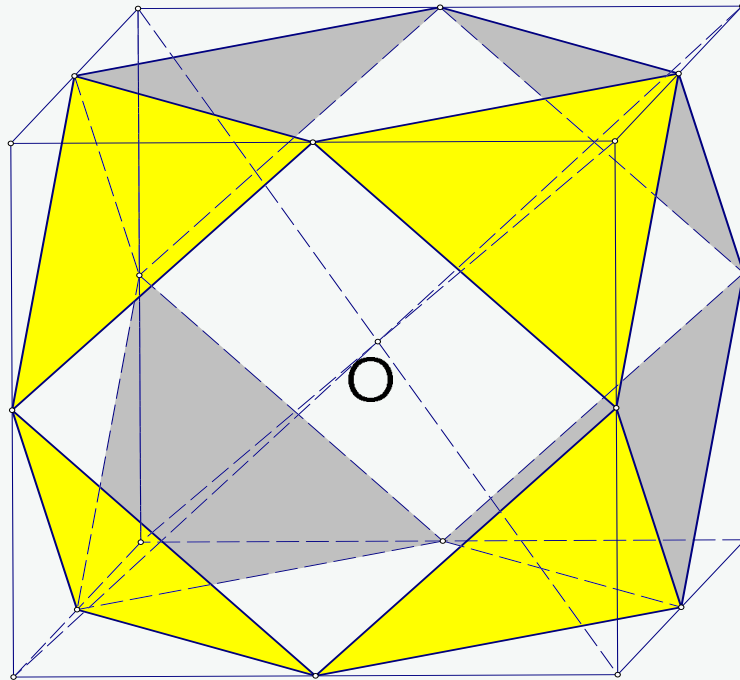
n	26	27	28	29	30	31	32
A103158	442587	530508	631820	1748121	880941	31031930	1202984





Why this pattern?

Cuboctahedron, $d = 1$



$$\boxed{1} \quad 1, 1, \{[0, 0, 0], [1, 1, 0], [1, 0, 1], [0, 1, 1]\}$$

$$N := \{[[-1, -1, 1], 1], [[1, -1, -1], 1], [[1, -1, 1], 1], [[-1, -1, -1], 1]\}$$

$$\boxed{2} \quad 3, 4, \{[1, 1, 0], [0, 0, 4], [4, 1, 3], [1, 4, 3]\}$$

$$N := \{[[-1, -1, 1], 1], [[-1, 5, 1], 3], [[-5, 1, -1], 3], [[1, 1, 5], 3]\}$$

$$\boxed{3} \quad 5, 7, \{[0, 0, 4], [7, 0, 3], [3, 5, 0], [4, 5, 7]\},$$

$$N := \{[[1, 5, 7], 5], [[7, -5, -1], 5], [[1, -5, 7], 5], [[7, 5, -1], 5]\}$$

$$\boxed{4} \quad 7, 9, \{[9, 0, 9], [0, 4, 8], [8, 9, 5], [5, 1, 0]\}$$

$$N := \{[[-5, -11, 1], 7], [[-1, 1, -1], 1], [[-1, -5, -11], 7], [[-11, 1, 5], 7]\}$$

$$\boxed{5} \quad 9, 12, \{[11, 9, 0], [11, 0, 9], [0, 5, 5], [8, 12, 12]\}$$

$$N := \{[[-1, -11, -11], 9], [[-7, 13, -5], 9], [[-5, -1, -1], 3], [[7, 5, -13], 9]\}$$

$$\boxed{6} \quad 11, 15, \{[4, 0, 0], [7, 13, 8], [15, 0, 11], [0, 1, 15]\}$$

$$N := \{[[13, 5, -13], 11], [[-1, -19, 1], 11], [[17, -7, 5], 11], [[-5, -7, -17], 11]\}$$

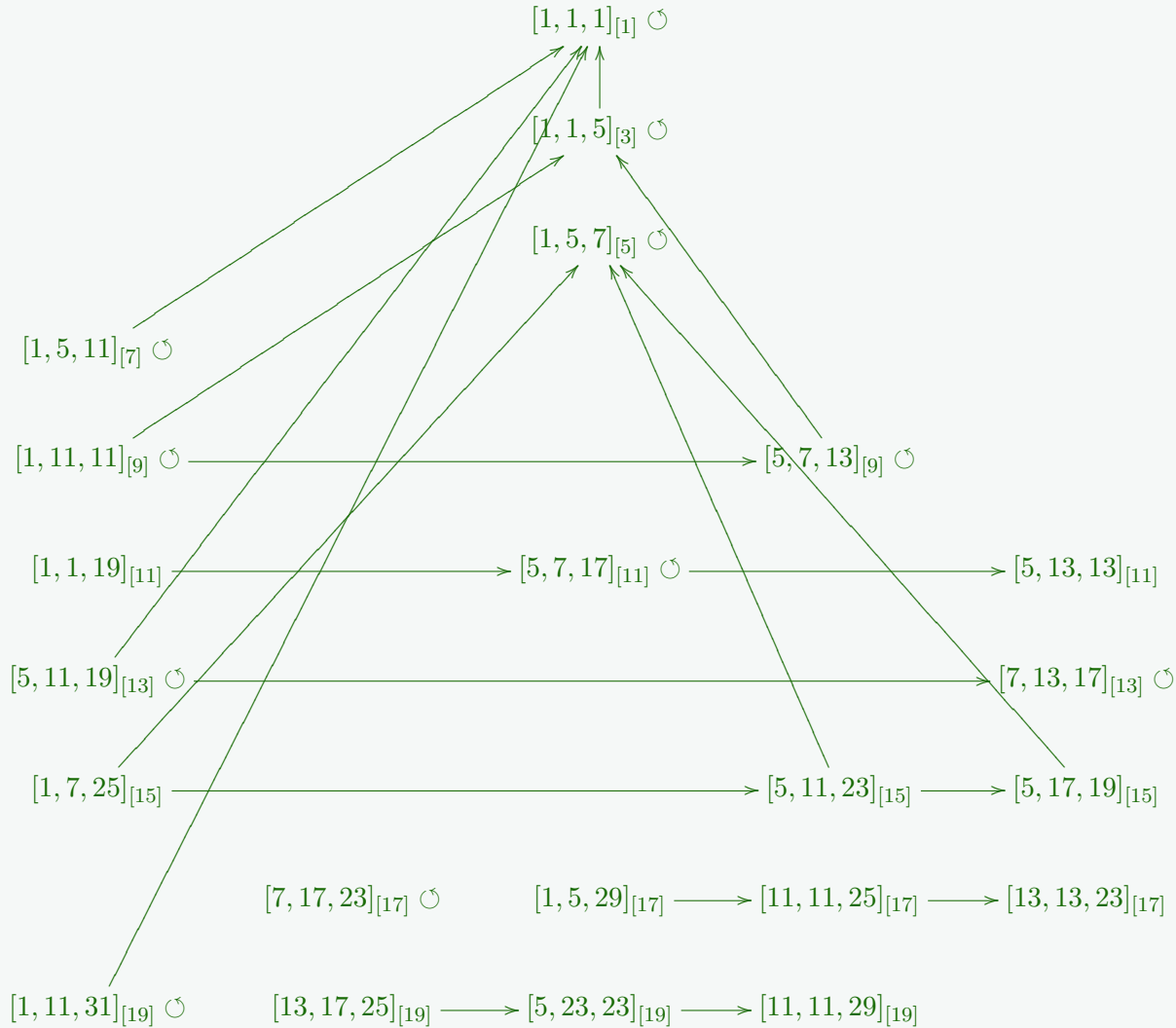
$$\boxed{7} \quad 13, 16, \{[0, 9, 15], [15, 16, 7], [16, 0, 16], [7, 1, 0], [5, 11, 19]\}$$

$$N := \{[[-5, -11, -19], 13], [[11, 19, -5], 13], [[19, -5, -11], 13], [[-1, 1, -1], 1]\}$$

$$\boxed{8} \quad 13, 17, \{[17, 13, 5], [0, 13, 12], [5, 0, 0], [12, 0, 17], [7, 13, 17]\}$$

$$N := \{[[-17, -13, 7], 13], [[7, -13, 17], 13], [[-17, 13, 7], 13], [[7, 13, 17], 13]\}$$

Graph RT



Example 1: The regular tetrahedron $OABC$ where $O = (0, 0, 0)$, $A = (376, -841, 2265)$, $B = (-1005, -2116, 701)$, $C = (1411, -1965, 356)$ has the four faces with normal vectors.

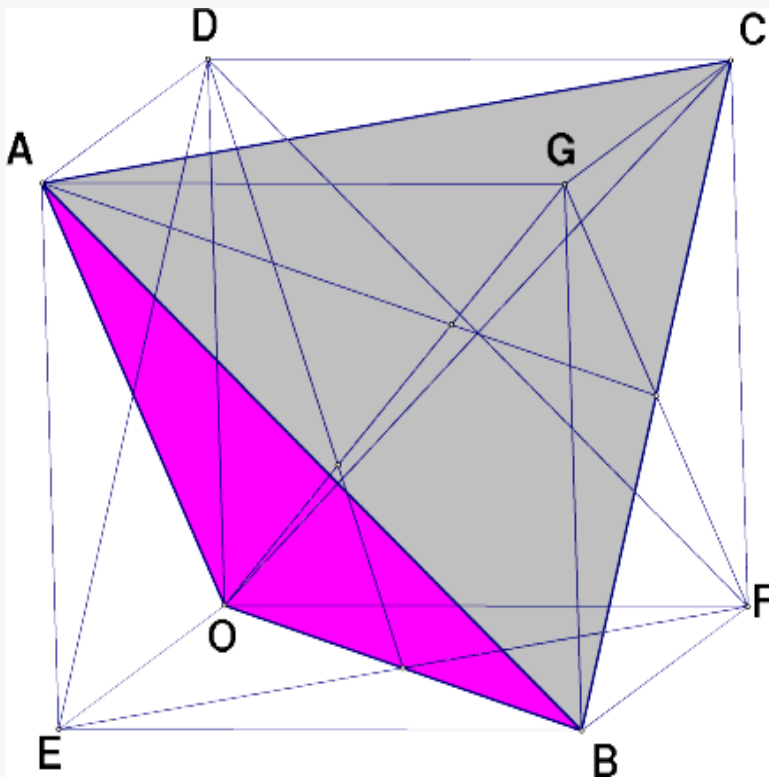
$$\begin{aligned} &(-187, 113, 73), \text{ satisfying } 187^2 + 113^2 + 73^2 = 3(133^2), \\ &(-343, -253, -37), \text{ satisfying } 343^2 + 253^2 + 37^2 = 3(247)^2, \\ &(19, 41, 151), \text{ satisfying } 19^2 + 41^2 + 151^2 = 3(91)^2 \text{ and} \\ &(391, -2461, 1661), \text{ satisfying } 391^2 + 2461^2 + 1661^2 = 3(1729)^2. \end{aligned}$$

Example 2: The points $O = (0, 0, 0)$, $A = [-6677, -2672, 1445]$, $B = [-5940, 4143, -1167]$, $C = [-3837, 2595, 5688]$ form a regular tetrahedron of side-lengths equal to $5187\sqrt{2}$ and the highest d for its faces is 1729.

Observation: $5187 = (3)(7)(13)(19)$ and the primes 3, 7, 13 and 19 are first primes of the form $x^2 + 3y^2$, $x, y \in \mathbb{Z}$.

The cubes in \mathbb{Z}^3

Theorem B *Every cube in \mathbb{Z}^3 can be obtained by a translation along a vector with integer coordinates from a cube with a vertex the origin containing a regular tetrahedron with a vertex at the origin and all integer coordinates (see figure below) and as a result it must have side lengths equal to n for some $n \in \mathbb{N}$. Conversely, given a regular tetrahedron in \mathbb{Z}^3 , this can be completed to a cube which is going to be automatically in \mathbb{Z}^3 .*



The number of cubes whose coordinates for its vertices are in the set $\{0, 1, \dots, n\}$ is the sequence A098928:

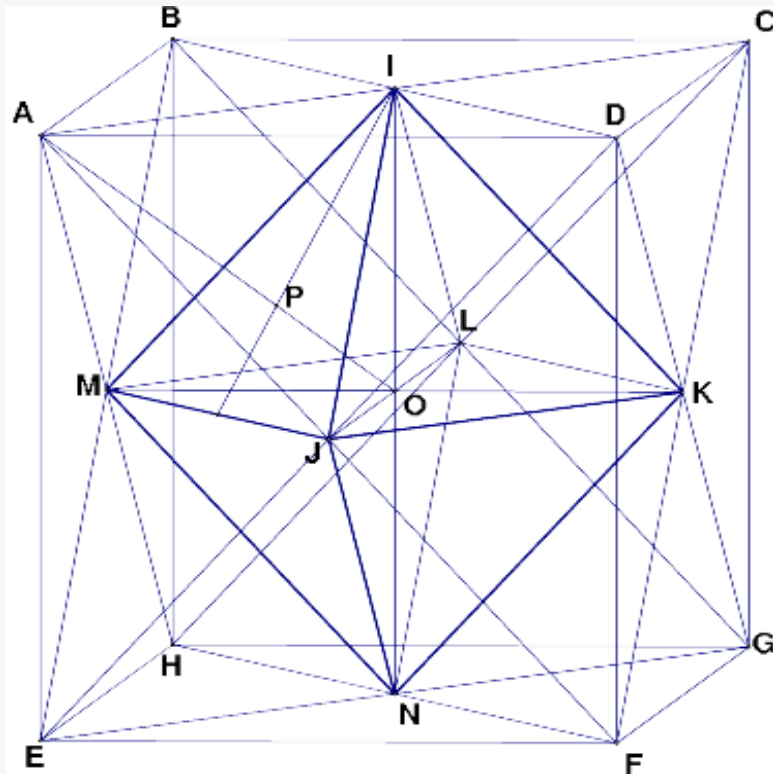
n	1	2	3	4	5	6	7	8	9	10	11
A098928	1	9	36	100	229	473	910	1648	2795	4469	6818

n	12	13	14	15	16	17	18
A098928	10032	14315	19907	27190	36502	48233	62803

$$A098928 \leq A103158$$

The octahedrons in \mathbb{Z}^3

Theorem C *Every regular octahedron in \mathbb{Z}^3 is the dual of a cube that can be obtained (up to a translation with a vector with integer coordinates) by doubling a cube in \mathbb{Z}^3 .*



The number of regular octahedrons whose coordinates for its vertices are in the set $\{0, 1, \dots, n\}$ denoted by $\mathcal{RO}(n)$:

n	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{RO}(n)$	0	1	8	32	104	261	544	1000	1696	2759	4296

n	12	13	14	15	16	17	18
$\mathcal{RO}(n)$	6434	9352	13243	18304	24774	32960	43223

Orthogonal matrices with rational coefficients

$$T_3 := \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{pmatrix}.$$

$$T_{2009} := \frac{1}{2009} \begin{pmatrix} 210 & 1645 & 1134 \\ -1330 & 966 & -1155 \\ 1491 & 630 & -1190 \end{pmatrix}.$$

Good source of finite subgroups $GL(3, \mathbb{Z}_p)$ by taking convenient primes.

Conjectures:

- ① *The Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ has degenerate solutions, i.e. $\gcd(a, b, c) = 1$, $\gcd(a, d) > 1$, $\gcd(b, d) > 1$ and $\gcd(c, d) > 1$, if and only if d has at least three distinct prime factors of the form $4k+1$, $k \in \mathbb{N}$.*
- ② *The graph RT has infinitely many connected components and a fractal structure.*
- ③ *The tetrahedron $OABC$, with $O = (0, 0, 0)$, $A = [-6677, -2672, 1445]$, $B = [-5940, 4143, -1167]$, $C = [-3837, 2595, 5688]$, gives the smallest side between all sides $\ell\sqrt{2}$ of irreducible regular tetrahedra with the property that their faces have equations $a_i^2 + b_i^2 + c_i^2 = 3d_i^2$ with $d_i < \ell$.*
- ④ *There is a way of finding a pair $(\vec{\zeta}, \vec{\eta})$ with symmetric formulae in terms of a , b and c .*

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