
k -Dependence and Domination in Kings Graphs*

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1. INTRODUCTION. Among combinatorial chessboard problems, the following [16] is one of the most basic:

What is the maximum number of kings that can be placed on an $m \times n$ board, so that no two squares occupied by kings share a side or a corner (i.e., no king “attacks” another)?

By placing kings in row i and column j when i and j are both odd, we see that at least $\lceil m/2 \rceil \lceil n/2 \rceil$ squares can be occupied by kings. To see that this is optimal, note that any 2×2 square can contain at most one king. For m and n large, one learns that the best one can do is to place kings on about $1/4$ of the squares. In this paper, we study the following general version of this problem:

Given a whole number $k \leq 8$ (8 being the maximum number of squares a king can attack), what is the maximum number s of kings that can be placed on an $m \times n$ board, so that no king attacks more than k other kings? When m and n are large, how large can the density $s/(mn)$ be?

For most choices of $k = 0, \dots, 8$, there is a tidy solution: an upper bound can be proved by a short elementary argument, and an arrangement of kings can be constructed to show that the upper bound is tight. These limiting densities are given in Section 6. However, tight upper bounds are not yet known for either $k = 4$ or $k = 5$. It is easy to construct arrangements of kings (on arbitrarily large boards) that achieve the densities of $3/5$ and $9/13$ for $k = 4$ and 5 , respectively. We conjecture that these are indeed the maximum limiting densities.

The story in the present article concerns the struggle to support this conjecture by good upper bounds, as well as the variety of rival techniques used for different values of k . Along the way, we make elementary use of graph theory, number theory, group theory, real analysis, and integer linear programming. We believe the methods of the present paper can provide the basis for undergraduate research projects on related problems.

2. NOTATION AND TERMINOLOGY. We have already deviated from traditional chess in several ways: the board’s length and width are arbitrary; each chess piece is a king with no associated color; we are concerned with optimal arrangements of pieces, rather than actual chess moves. We actually go a few steps further. First, we generalize the discussion to address the density problem of placing kings on multidimensional chessboards. Second, it is useful to also treat *toroidal* boards allowing “wrap-around”; these provide an idealization with the same limiting densities as nontoroidal boards, but with a simpler analysis. Third, some results are stated in terms of arbitrary graphs. These three extensions also serve to identify possible areas for undergraduate research.

*or “Too Many Kings and There Goes the Neighborhood”

We adopt notation and terminology from graph theory by referring to board squares as *vertices*. We let $K[n_1, \dots, n_d]$ denote the $n_1 \times \dots \times n_d$ *kings graph* whose vertex set is the Cartesian product $[n_1] \times \dots \times [n_d]$, where $[n]$ denotes $\{1, 2, \dots, n\}$. Two vertices are called *neighbors* (or said to be *adjacent*) when we can get from one to the other by a single generalized king's move. In other words, distinct vertices $\mathbf{v} = (v_1, \dots, v_d)$ and $\mathbf{u} = (u_1, \dots, u_d)$ in $K[n_1, \dots, n_d]$ are neighbors if and only if $|v_i - u_i| \leq 1$ for each $i \in [d]$.

We define the *toroidal kings graph* $K_{\text{tor}}[n_1, \dots, n_d]$ on the same vertex set as $K[n_1, \dots, n_d]$, but consider distinct vertices \mathbf{v} and \mathbf{u} to be neighbors in $K_{\text{tor}}[n_1, \dots, n_d]$ if and only if $v_i - u_i \equiv -1, 0, \text{ or } 1 \pmod{n_i}$ for each $i \in [d]$. The analysis of $K_{\text{tor}}[n_1, \dots, n_d]$ is much simpler than that of $K[n_1, \dots, n_d]$ because vertices of $K_{\text{tor}}[n_1, \dots, n_d]$ have equally many neighbors.¹

Now let G be an arbitrary (loopless) graph with vertex set $V(G)$. For a vertex $\mathbf{v} \in V(G)$, $N(\mathbf{v})$ denotes the set of vertices adjacent to \mathbf{v} . We call $N(\mathbf{v})$ the *neighborhood* of \mathbf{v} in G , noting that $N(\mathbf{v})$ does not include the vertex \mathbf{v} itself. Next, consider a whole number k and a set $S \subseteq V(G)$. As introduced by Fink and Jacobson [5], we say that S is *k-dependent* in G if $|N(\mathbf{v}) \cap S| \leq k$ for each $\mathbf{v} \in S$, so that each vertex of S has at most k neighbors in S . The name “ k -dependent” arises from the case $k = 0$, since a 0-dependent set corresponds to an *independent* set in graph theory. The *k-dependence number* of G , denoted by $\beta_k(G)$, is the maximum cardinality of a k -dependent set in G .

For a k -dependent set S in a kings graph (toroidal or otherwise), we regard S as the set of vertices or squares occupied by kings, no king having more than k neighboring kings. For example, Figure 1(a) shows a 4-dependent set of 43 kings (indicated by dark squares) arranged in $K_{\text{tor}}[6, 12]$, proving that $\beta_4(K_{\text{tor}}[6, 12]) \geq 43$. Likewise, Figure 1(b) shows a 5-dependent set of 117 kings in $K_{\text{tor}}[13, 13]$, demonstrating that $\beta_5(K_{\text{tor}}[13, 13]) \geq 117$.

Part of our motivation comes from [2], which includes a section on “1/2-dominating” of kings graphs $K[m, n]$ for small values of m . A subset R of $V(G)$ is a *1/2-dominating set* if each vertex \mathbf{v} of $S = V(G) \setminus R$ satisfies $|R \cap N(\mathbf{v})| \geq |N(\mathbf{v})|/2$; so, assuming $N(\mathbf{v}) \neq \emptyset$ for each vertex \mathbf{v} , R is a *dominating set* with the additional feature that each vertex not in R is dominated by at least half of its neighbors. With our emphasis on k -dependence, we take the complementary perspective, defining a subset S of $V(G)$ as *half-dependent* in G if each vertex \mathbf{v} of S satisfies $|S \cap N(\mathbf{v})| \leq |N(\mathbf{v})|/2$. The *half-dependence number*, denoted by $h(G)$, is the maximum cardinality among half-dependent sets in G . The dark squares in Figure 1(c) form a half-dependent set of 694 vertices in $K[34, 34]$, the white squares a 1/2-dominating set, from which we infer that $h(K[34, 34]) \geq 694$.

For a graph G , we let $\rho_k(G)$ denote $\beta_k(G)/|V(G)|$, the maximum *density* among k -dependent sets in G . For a given dimension d , $K^{(d)}[n]$ and $K_{\text{tor}}^{(d)}[n]$ denote the special cases of $K[n_1, \dots, n_d]$ and $K_{\text{tor}}[n_1, \dots, n_d]$, respectively, in which $n_i = n$ for each i .

¹Note, however, that toroidal chessboards for which some $n_i < 3$ generally require separate handling. In particular, the effects of adding 1 or -1 in coordinate i are precisely the same when $n_i = 2$. For example, in the toroidal board to be represented by $K_{\text{tor}}[2, 8, 2]$, there are four ways to move from vertex $(2, 6, 1)$ to $(1, 7, 2)$: simply add any of the vectors $(1, 1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, $(-1, 1, -1)$ to $(2, 6, 1)$. Thus $K_{\text{tor}}[2, 8, 2]$ should be defined as a *multigraph*, in the sense that these two vertices are “neighbors of multiplicity 4.” In $K_{\text{tor}}[n_1, \dots, n_d]$ this *multiplicity* is 2^c , where c is the number of coordinates i at which two neighbors differ and for which $n_i = 2$. By counting multiplicities, all of our results can easily be extended to cover this situation, so we give it no further special treatment. Note that when $n_i = 1$, each vertex is a multiple neighbor of itself; however, removing index i leads to an equivalent problem in lower dimensions, so we assume $n_i > 1$ in this article.

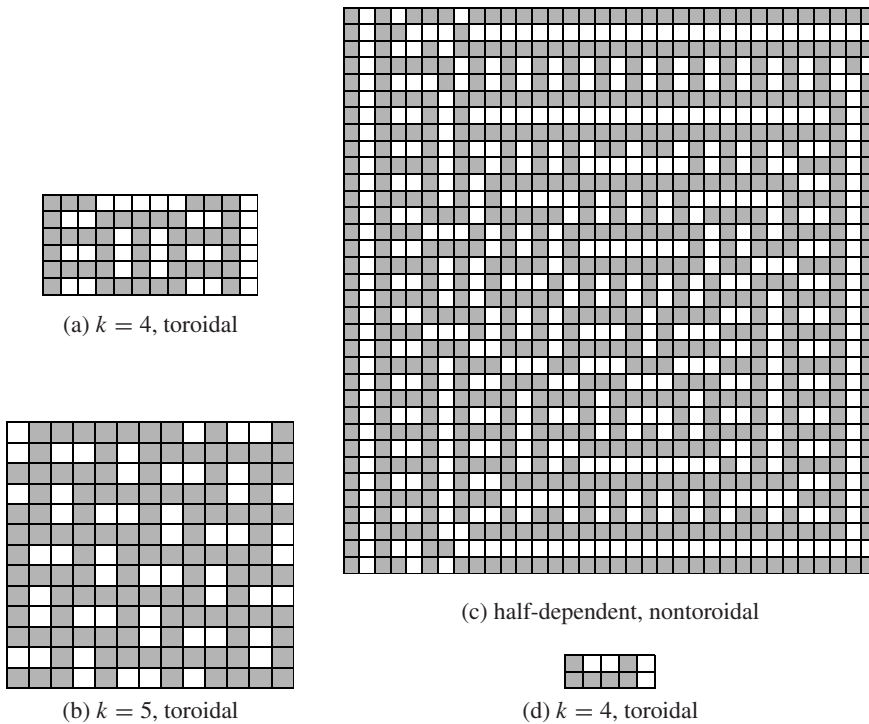


Figure 1. Examples of k -dependent and half-dependent sets.

As a further shorthand, we use the following:

$$\rho_k^{(d)}(n) = \rho_k(K_{\text{tor}}^{(d)}[n]), \quad \rho_k^{(d)} = \lim_{n \rightarrow \infty} \rho_k^{(d)}(n).$$

In this paper, we prove that the limit $\rho_k^{(d)}$ exists and we seek its exact value. We provide good upper and lower bounds in many cases, and we obtain exact values for $\rho_k^{(2)}$ when $k \neq 4, 5$. Based on our results, we suspect that $\rho_k^{(d)}$ is a rational number for any d and k .

For more on combinatorial chessboard problems see [1], [4], [6], [9], [15], [14]; for k -dependence see [3], [7]; and for similar problems see [8], [13].

3. TWO-DIMENSIONAL KINGS GRAPHS. The original motivation for this paper concerned the following conjecture about the maximum density of kings on a two-dimensional board.

Conjecture 1. *For two-dimensional boards, the half-dependent limiting density is*

$$\lim_{n \rightarrow \infty} \frac{h(K[n, n])}{|V(K[n, n])|} = 3/5,$$

and the half-dependent sets in a kings graph satisfy

$$h(K[n, n]) \geq \frac{3n^2}{5} - C,$$

for all n and some constant C .

Through direct construction, we have verified that $h(K[n, n]) \geq (3n^2/5) - (7/5)$ for each $n \leq 35$. Examples for $n = 3, \dots, 11$ are shown in Figure 2, whereas Figure 1(c) provides a half-dependent set of 694 kings, thereby demonstrating $h(K[n, n]) \geq (3n^2/5) - 2$ in the case $n = 34$. In a later section, we give more credence to Conjecture 1 by establishing the upper bound $\lim_{n \rightarrow \infty} h(K[n, n])/|V(K[n, n])| \leq 0.608956$. Meanwhile, we can give some lower bounds on $h(K[n, n])$.

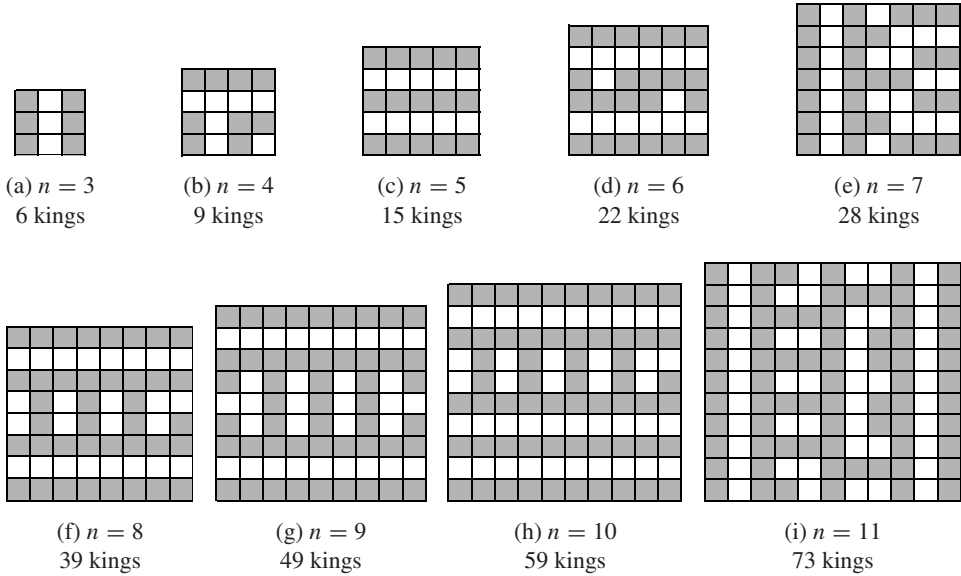


Figure 2. Examples of maximum-density, half-dependent sets.

Theorem 1. *The maximum size of a half-dependent set in the $n \times n$ kings graph satisfies the following lower bounds, for some constant C :*

- $h(K[n, n]) \geq \frac{3}{5}n^2 - \frac{n}{30} - C$, if $n \equiv 0 \pmod{5}$;
- $h(K[n, n]) \geq \frac{3}{5}n^2 - \frac{4n}{30} - C$, if $n \equiv 1 \pmod{5}$;
- $h(K[n, n]) \geq \frac{3}{5}n^2 - \frac{2n}{30} - C$, if $n \equiv 2 \pmod{5}$;
- $h(K[n, n]) \geq \frac{3}{5}n^2 + \frac{3}{5}$, if $n \equiv 3 \pmod{5}$;
- $h(K[n, n]) \geq \frac{3}{5}n^2 - \frac{3n}{30} - C$, if $n \equiv 4 \pmod{5}$.

Proof. Let \mathcal{C} and \mathcal{D} denote the toroidal arrangements in Figures 1(d) and 1(a), respectively. Upon stacking m copies of \mathcal{C} and removing the king at $(2m, 3)$, we obtain an arrangement \mathcal{A}' comprising $6m - 1$ kings in $K[2m, 5]$. (The reader should note that all illustrations in this paper follow matrix indexing, so that vertex (i, j) appears as a square in row i , column j .) In $K[2m + 1, 5]$ upon placing a copy of \mathcal{A}' in $\{2, 3, \dots, 2m + 1\} \times [5]$ and including additional kings at $(1, 1)$, $(1, 2)$, and $(1, 4)$, we have an arrangement \mathcal{A}'' comprising $6m + 2$ kings. Thus via \mathcal{A}' or \mathcal{A}'' we have an arrangement \mathcal{A} in $K[n, 5]$ comprising $3n - 1$ kings, using no kings in column 5.

Similarly, stacking $m = \lfloor (n - 2)/6 \rfloor$ copies of \mathcal{D} and placing the result within $\{2, 3, \dots, 6m + 1\} \times [12]$ we obtain an arrangement \mathcal{B} in $K[n, 12]$ consisting of $43m = (43/72)(12n) - c = (43n/6) - c$ kings for some constant c (based on the fact that we have generously left row 1 and rows $6m + 2$ through n devoid of kings). Arrangement \mathcal{B} has no kings in column 12.

We now use \mathcal{A} and \mathcal{B} to construct the desired half-dependent arrangements within $K[n, n]$. If $n = 5m + 3$ then place m copies of \mathcal{A} side by side in $[n] \times \{3, 4, \dots, n - 1\}$ and kings everywhere in columns 1 and n to verify the result, as in Figure 2(f) when $n = 8$. If $n = 5m + 15$ then place m copies of \mathcal{A} side by side in $[n] \times \{3, 4, \dots, n - 13\}$ followed by a copy of \mathcal{B} in columns $n - 12$ through $n - 1$ and kings everywhere in columns 1 and n . Similarly, use two copies of \mathcal{B} if $n = 5m + 27$, three copies of \mathcal{B} if $n = 5m + 39$, and four copies of \mathcal{B} if $n = 5m + 51$. The result is thus verified for large n in each congruence case, and small values of n are automatically correct by specifying the constant C sufficiently large in compensation. ■

4. LIMITING DENSITIES. Our next result shows that the limiting densities exist and illustrates the tight relationship between the half-dependent nontoroidal and k -dependent toroidal problems. It also provides a lower bound on the limiting densities.

Theorem 2. *The limiting values*

$$\rho_k^{(d)} = \lim_{n \rightarrow \infty} \frac{\beta_k(K_{\text{tor}}^{(d)}[n])}{n^d} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{h(K^{(d)}[n])}{n^d}$$

exist and satisfy

- (a) $\rho_k^{(d)} \geq \rho_k(K_{\text{tor}}[n_1, \dots, n_d])$, for any choice of d, k , and n_1, \dots, n_d ;
- (b) $\rho_k^{(d)} = \lim_{n \rightarrow \infty} h(K^{(d)}[n])/n^d$, in the case where $k = (3^d - 1)/2$.

Proof. Consider $n_1, \dots, n_d > 0$ and $n > \max_i n_i$. The quotient-remainder theorem (i.e., division algorithm) allows us to uniquely write $n = n_i \lfloor n/n_i \rfloor + r_i$ for some $r_i \in \{0, \dots, n_i - 1\}$. We can then pack $K_{\text{tor}}^{(d)}[n]$ with $\prod_{i=1}^d \lfloor n/n_i \rfloor$ nonoverlapping copies of $K_{\text{tor}}[n_1, \dots, n_d]$. These copies can be aligned so that the toroidal boundaries are compatible from one copy to the next, except for those abutting the “remainder” sections of length r_i in each coordinate. Figure 3 illustrates such a packing.

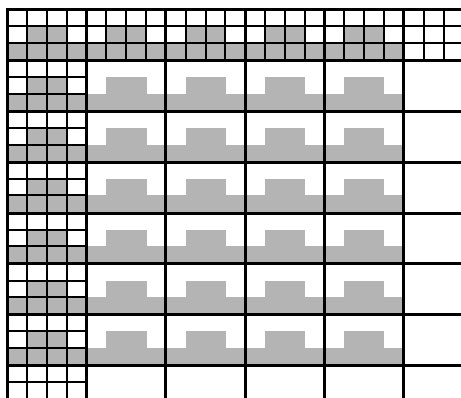


Figure 3. A packing of $K_{\text{tor}}[23, 23]$ with $\lfloor 23/3 \rfloor \cdot \lfloor 23/4 \rfloor$ copies of $K_{\text{tor}}[3, 4]$, each copy containing a 4-dependent set of maximum density.

Next, we place a k -dependent set of density $\rho_k(K_{\text{tor}}[n_1, \dots, n_d])$ within each copy of $K_{\text{tor}}[n_1, \dots, n_d]$. This yields a k -dependent set S in $K_{\text{tor}}^{(d)}[n]$, thereby giving us the bound

$$\begin{aligned} \rho_k^{(d)}(n) &\geq \frac{|S|}{n^d} = \frac{\beta_k(K_{\text{tor}}[n_1, \dots, n_d]) \prod_{i=1}^d \lfloor n/n_i \rfloor}{n^d} \\ &= \rho_k(K_{\text{tor}}[n_1, \dots, n_d]) \prod_{i=1}^d \frac{\lfloor n/n_i \rfloor}{n/n_i}. \end{aligned} \quad (1)$$

In the special case where $n_i = m < n$ for all i , this implies

$$\rho_k^{(d)}(n) \geq \rho_k^{(d)}(m) \left(\frac{\lfloor n/m \rfloor}{n/m} \right)^d.$$

Taking the limit infimum as $n \rightarrow \infty$ yields

$$\liminf_{n \rightarrow \infty} \rho_k^{(d)}(n) \geq \rho_k^{(d)}(m), \quad \forall m > 0. \quad (2)$$

From here, the limit supremum as $m \rightarrow \infty$ gives us

$$\liminf_{n \rightarrow \infty} \rho_k^{(d)}(n) \geq \limsup_{m \rightarrow \infty} \rho_k^{(d)}(m).$$

Consequently, $\rho_k^{(d)} = \lim_{n \rightarrow \infty} \rho_k^{(d)}(n)$ exists. Combined with inequality (1), this also proves statement (a).

Now let $k = (3^d - 1)/2$. By deleting the ‘‘boundary’’ kings from a k -dependent subset of density $\rho_k^{(d)}(n)$ on the toroidal board $K_{\text{tor}}^{(d)}[n]$, we obtain a half-dependent subset S of the nontoroidal board $K^{(d)}[n]$. This implies that

$$\frac{h(K^{(d)}[n])}{n^d} \geq \frac{|S|}{n^d} \geq \rho_k^{(d)}(n) - \frac{n^d - (n-2)^d}{n^d}.$$

Reversing the roles of the two boards yields the analogous inequality

$$\rho_k^{(d)}(n) \geq \frac{h(K^{(d)}[n])}{n^d} - \frac{n^d - (n-2)^d}{n^d}.$$

Combining these and taking the limit proves statement (b) and the existence of $\lim_{n \rightarrow \infty} h(K^{(d)}[n])/n^d$. ■

The analogue of Theorem 2(a) in which $\rho_k^{(d)}$ and $\rho_k(K_{\text{tor}}[n_1, \dots, n_d])$ are replaced by their half-dependent counterparts on nontoroidal boards fails:

$$\lim_{n \rightarrow \infty} \frac{h(K^{(2)}[n])}{n^2} \leq 0.609 < \frac{h(K^{(2)}[3])}{3^2} = \frac{2}{3},$$

as demonstrated by Theorems 1 and 4.

Figure 1(d) shows a 4-dependent set of 6 kings in $K_{\text{tor}}[2, 5]$, where each king has four neighboring kings (when the neighbors are counted with multiplicity). Combining Theorem 2 with the examples of Figures 1(b) and 1(d), we have verified the following numerical lower bounds.

Corollary 1. *For 4- and 5-dependent kings graphs in two dimensions, we have the lower bounds $\rho_4^{(2)} \geq 3/5$ and $\rho_5^{(2)} \geq 9/13$.*

5. BINARY LINEAR PROGRAMMING. The k -dependence number $\beta_k(G)$ of a given graph G can be computed, in principle, by reformulating the corresponding maximization problem. To each k -dependent set S , we associate the *characteristic vector* with entries

$$x_v = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

To each vertex $v \in V(G)$, we associate the inequality

$$(|N(v)| - k)x_v + \sum_{u \in N(v)} x_u \leq |N(v)| \tag{3}$$

and impose the restriction that x_v be 0 or 1. (In the case of a multigraph, such as a toroidal kings graph for which some $n_i = 2$, the summation term in (3) must be modified to account for multiplicities.) Separate consideration of the cases $x_v = 0$ and $x_v = 1$ shows that the resulting system of inequality constraints precisely describes k -dependent sets. The optimization problem consists of maximizing the linear function $\sum_{v \in V(G)} x_v$ subject to this system of inequalities and the 0-1 restrictions. This is an example of a *binary linear programming* (binary LP) problem. This problem's optimal value is $\beta_k(G)$, with optimal solutions corresponding to k -dependent sets of maximum cardinality.

The optimization problem for determining $h(G)$ can be formulated similarly, except that the constraint associated with each vertex v becomes

$$\left\lceil \frac{|N(v)|}{2} \right\rceil x_v + \sum_{u \in N(v)} x_u \leq |N(v)|.$$

Applying a binary LP solver to this problem, we determined the values of $h(K[n, n])$ shown in Table 1 for $1 \leq n \leq 11$, along with the sample half-dependent sets of $h(K[n, n])$ kings shown in Figure 2. For $n = 8, 9, 10, 11$, all optimal patterns look like the samples shown; for $n = 7$, there are several distinct optimal patterns, including a pattern consisting of vertical stripes.

Table 1. Maximum number of kings in half-dependent sets.

n	1	2	3	4	5	6	7	8	9	10	11
$h(K[n, n])$	1	2	6	9	15	22	28	39	49	59	73

The “binary” restriction of these LPs (namely, that x_v be 0 or 1) makes it possible to search through all $2^{|V(G)|}$ arrangements, looking for a largest k -dependent one. Although modern software for solving LPs manages to avoid considering nearly so many arrangements, there quickly comes a point where $|V(G)|$ is simply too large for this method to be practical. The next section shows how the binary LP perspective can still allow for efficient calculation of good upper bounds.

6. UPPER BOUNDS AND EXACT VALUES. In this section, we calculate upper bounds by solving binary LPs on relatively small vertex sets. In fact, the system of inequalities derived in the preceding section can lead to general upper bounds without

even solving the associated linear program! As an example, consider the problem of calculating $\beta_2(K_{\text{tor}}[n, n])$, the maximum number of kings that can be placed on an $n \times n$ toroidal board with no king having more than two neighboring kings. In this case, the LP constraint (3) is

$$6x_{\mathbf{v}} + \sum_{\mathbf{u} \in N(\mathbf{v})} x_{\mathbf{u}} \leq 8.$$

Note that $x_{\mathbf{v}}$ appears in nine of these n^2 constraints: once (with coefficient 6) in its own associated constraint and once (with coefficient 1) in each of the eight constraints associated with its neighbors. Summing the constraints over \mathbf{v} , we find that the characteristic vector of a 2-dependent set S satisfies

$$14|S| = 14 \sum_{\mathbf{v} \in V(G)} x_{\mathbf{v}} \leq 8n^2.$$

Thus $|S| \leq (4/7)n^2$, which implies that $\rho_2^{(2)}(n) \leq 4/7$.

But we can do better. Observe that, for any 2-dependent set S and any vertex \mathbf{v} in an $n \times n$ toroidal board, we have $\sum_{\mathbf{u} \in N(\mathbf{v})} x_{\mathbf{u}} \leq 6$, since placing kings at 7 of the 8 neighbors of \mathbf{v} always violates 2-dependence. By once again considering the cases $x_{\mathbf{v}} = 0$ and $x_{\mathbf{v}} = 1$ separately, we see that the LP constraint associated with \mathbf{v} can be replaced by

$$4x_{\mathbf{v}} + \sum_{\mathbf{u} \in N(\mathbf{v})} x_{\mathbf{u}} \leq 6.$$

Summing this new set of constraints yields $12|S| \leq 6n^2$, thereby improving the bound to $\rho_2^{(2)}(n) \leq 1/2$. On the other hand, we can form a 2-dependent set by placing a king at $\mathbf{v} = (v_1, v_2)$ if and only if v_2 is even. This shows that $\beta_2(K_{\text{tor}}[n, n]) \geq n \lfloor n/2 \rfloor$ and therefore $\lfloor n/2 \rfloor / n \leq \rho_2^{(2)}(n) \leq 1/2$. In the limit, we obtain $\rho_2^{(2)} = 1/2$. There are simpler ways to obtain the exact value for $\rho_2^{(2)}$, but the approach just given can be generalized, as we show next.

An *automorphism* of a graph G is a bijection $f : V(G) \rightarrow V(G)$ that preserves adjacency, so that neighbors are mapped to neighbors (and nonneighbors are mapped to nonneighbors). A graph G is *vertex-transitive* if for every two vertices \mathbf{v}, \mathbf{u} there exists an automorphism f for which $f(\mathbf{v}) = \mathbf{u}$. In other words, G is vertex-transitive when each vertex plays the same structural role in G as any other vertex, such as happens in toroidal kings graphs but not in kings graphs. Note that in a vertex-transitive graph, the neighborhoods $N(\mathbf{v})$ all have the same cardinality. In the following result, $\langle V' \rangle$ denotes the subgraph *induced* by a subset V' of $V(G)$, namely, the subgraph of G formed by deleting all vertices of G not in V' .

Proposition 1. *Suppose G is a vertex-transitive graph and let β_k^* denote the quantity $\beta_k(\langle N(\mathbf{v}) \rangle)$, which is independent of the choice of vertex \mathbf{v} . Then*

$$\beta_k(G) \leq \frac{\beta_k^* |V(G)|}{\beta_k^* - k + |N(\mathbf{v})|}.$$

Proof. Associate a constraint $(\beta_k^* - k)x_{\mathbf{v}} + \sum_{\mathbf{u} \in N(\mathbf{v})} x_{\mathbf{u}} \leq \beta_k^*$ to each vertex $\mathbf{v} \in V(G)$, sum the constraints over all \mathbf{v} , and deduce the maximum of $|S| = \sum_{\mathbf{v} \in V(G)} x_{\mathbf{v}}$. ■

As an application of Proposition 1, we return to the two-dimensional setting of arranging kings on a large $n \times n$ board. In this case, the values of β_k^* for $G = K_{\text{tor}}^{(2)}[n]$ are easily calculated by hand:

$$\beta_0^* = \beta_1^* = 4, \quad \beta_2^* = \beta_3^* = 6, \quad \beta_4^* = \beta_5^* = \beta_6^* = \beta_7^* = \beta_8^* = 8.$$

The corresponding upper bounds on $\rho_k^{(2)}$ are

$$\begin{aligned} \rho_0^{(2)} &\leq \frac{4}{12}, & \rho_1^{(2)} &\leq \frac{4}{11}, & \rho_2^{(2)} &\leq \frac{6}{12}, & \rho_3^{(2)} &\leq \frac{6}{11}, & \rho_4^{(2)} &\leq \frac{8}{12}, \\ \rho_5^{(2)} &\leq \frac{8}{11}, & \rho_6^{(2)} &\leq \frac{8}{10}, & \rho_7^{(2)} &\leq \frac{8}{9}, & \rho_8^{(2)} &\leq \frac{8}{8}, \end{aligned}$$

whereas the exact values for six of these turn out to be

$$\rho_0^{(2)} = \frac{1}{4}, \quad \rho_1^{(2)} = \frac{1}{3}, \quad \rho_2^{(2)} = \rho_3^{(2)} = \frac{1}{2}, \quad \rho_6^{(2)} = \frac{4}{5}, \quad \rho_7^{(2)} = \frac{8}{9}, \quad \rho_8^{(2)} = 1.$$

Thus, in these cases the upper bound of Proposition 1 is tight for $k \in \{0, 2, 6, 7, 8\}$. We can verify this tightness on a case-by-case basis. Note that $\rho_2^{(2)} = 1/2$ was proved earlier in this section. The values of $\rho_k^{(2)}$ with $k \in \{0, 7, 8\}$ correspond to taking $d = 2$ in the more general formulas

$$\rho_0^{(d)} = 2^{-d}, \quad \rho_{3^d-2}^{(d)} = 1 - 3^{-d}, \quad \rho_{3^d-1}^{(d)} = 1,$$

which can be established easily.

For $k = 6$, note that placing a king at vertex (i, j) in $K_{\text{tor}}[5, 5]$ when $3i + j \not\equiv 0 \pmod{5}$ forms a 6-dependent set. So, by Theorem 2(a), the upper bound $\rho_6^{(2)} \leq 4/5$ is tight. This case also generalizes to higher dimensions; see Section 7.

The upper bound in Proposition 1 is not tight for two-dimensional kings graphs with $k \in \{1, 3, 4, 5\}$, so additional methods are needed. In Section 8, we generalize Proposition 1 in a way that improves the upper bound in these four cases. The exact values for $\rho_1^{(2)}$ and $\rho_3^{(2)}$ are obtained by other means in Section 9.

7. LINEAR CONGRUENCES AND LOWER BOUNDS. Here we use linear congruences to produce specific k -dependent sets in $K_{\text{tor}}^{(d)}[n]$, thus giving lower bounds for $\rho_k^{(d)}$. These will match the upper bounds derived in the preceding section. For this purpose, we define a “modulo n ” remainder function

$$\mu_n(j) = j - n \lfloor j/n \rfloor.$$

For a vector $\mathbf{c} \in \{0, \dots, n-1\}^d$ and a set $R \subseteq \{0, \dots, n-1\}$, let $S(n, d, \mathbf{c}, R)$ denote the vertex set

$$S(n, d, \mathbf{c}, R) = \{\mathbf{v} \in V(K_{\text{tor}}^{(d)}[n]) : \mu_n(\mathbf{c} \cdot \mathbf{v}) \in R\}.$$

For which values of k is $S(n, d, \mathbf{c}, R)$ a k -dependent set? Consider a vertex $\mathbf{v} \in S(n, d, \mathbf{c}, R)$ and a nontrivial vector $\mathbf{y} \in \{-1, 0, 1\}^d$. Note that the neighbor $\mathbf{v} + \mathbf{y} \pmod{n}$ of \mathbf{v} belongs to $S(n, d, \mathbf{c}, R)$ if and only if $\mu_n(\mathbf{c} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{y}) \in R$. Thus, if we let $f(\mathbf{c}, r)$ denote the number of nontrivial vectors $\mathbf{y} \in \{-1, 0, 1\}^d$ for which $\mu_n(\mathbf{c} \cdot \mathbf{y} + r) \in R$, then $f(\mathbf{c}, \mathbf{c} \cdot \mathbf{v})$ is the number of elements of $S(n, d, \mathbf{c}, R)$ neighboring \mathbf{v} . We therefore have the following:

Lemma 1. For the choice $k = \max_{r \in R} f(\mathbf{c}, r)$, the set $S(n, d, \mathbf{c}, R)$ is k -dependent in $K_{\text{tor}}^{(d)}[n]$.

To obtain exact values of maximum toroidal densities in certain cases, we can combine Lemma 1 with the following well-known and easily verified fact: to each integer $m \neq 0$ there corresponds exactly one positive integer d and one vector $\mathbf{y} \in \{-1, 0, 1\}^d$ for which $m = \mathbf{c} \cdot \mathbf{y}$ and the first coordinate of \mathbf{y} is nonzero, where $\mathbf{c} = (3^{d-1}, 3^{d-2}, \dots, 3^1, 3^0)$. Such a representation of m as $\mathbf{c} \cdot \mathbf{y}$ corresponds to *balanced ternary notation*,² and leads to the following result on densities.

Theorem 3. When $k = 3^d - 3$, we have

$$\rho_k^{(d)} = \frac{3^d - 1}{3^d + 1}.$$

In particular, $\rho_6^{(2)} = 4/5$ and $\rho_{24}^{(3)} = 13/14$.

Proof. Consider $S = S(n, d, \mathbf{c}, R)$ with $n = (3^d + 1)/2$, $R = \{1, \dots, n - 1\}$, and

$$\mathbf{c} = (3^{d-1}, 3^{d-2}, \dots, 3^1, 3^0),$$

as in Lemma 1. It is straightforward to verify that, for any $r \in R$, there exist nontrivial vectors $\mathbf{y}', \mathbf{y}'' \in \{-1, 0, 1\}^d$ for which $\mathbf{c} \cdot \mathbf{y}' = -r$ and $\mathbf{c} \cdot \mathbf{y}'' = n - r$. Thus $f(\mathbf{c}, r) \leq 3^d - 3$, since at least two of the $3^d - 1$ choices of nontrivial vectors $\mathbf{y} \in \{-1, 0, 1\}^d$ must have $\mathbf{c} \cdot \mathbf{y} + r \notin R$. Therefore, S is $(3^d - 3)$ -dependent.

Now observe that, for each choice of values v_1, v_2, \dots, v_{d-1} , there is exactly one choice of $v_d \in \{0, \dots, n - 1\}$ for which $\mathbf{v} = (v_1, v_2, \dots, v_d) \notin S$. Consequently,

$$\frac{|S|}{|V(K_{\text{tor}}^{(d)}[n])|} = \frac{n - 1}{n} = \frac{3^d - 1}{3^d + 1},$$

so Theorem 2(a) gives us

$$\rho_k^{(d)} \geq \rho_k(K_{\text{tor}}^{(d)}[n]) \geq \frac{3^d - 1}{3^d + 1}.$$

For the matching upper bound, we apply Proposition 1 with $G = K_{\text{tor}}^{(d)}[n]$, $k = 3^d - 3$, and $|N(v)| = 3^d - 1$ to get

$$\begin{aligned} \rho_k^{(d)}(n) &= \frac{\beta_k(G)}{|V(G)|} \leq \frac{\beta_k^*}{\beta_k^* - k + |N(v)|} \\ &\leq \frac{3^d - 1}{(3^d - 1) - (3^d - 3) + (3^d - 1)} \\ &= \frac{3^d - 1}{3^d + 1}, \end{aligned}$$

where we use the fact that $\beta_k^* \leq |N(v)|$. Therefore $\rho_k^{(d)} \leq (3^d - 1)/(3^d + 1)$. ■

²For instance, corresponding to $m = 19$ are the choices $d = 4$ and $\mathbf{y} = (1, -1, 0, 1)$, based on the fact that 19 is expressible as $27 - 9 + 0 + 1 = (1)3^3 + (-1)3^2 + (0)3^1 + (1)3^0$.

8. IMPROVING UPPER BOUNDS FOR $\rho_4^{(2)}$ AND $\rho_5^{(2)}$. We now generalize the binary LP approach of Section 6 to address the unsolved problems of determining values for $\rho_4^{(2)}$ and $\rho_5^{(2)}$. Recall that in Proposition 1 we used the inequality $(\beta_k^* - k)x_v + \sum_{u \in N(v)} x_u \leq \beta_k^*$, which is valid for any k -dependent set in a vertex-transitive graph. Here we seek other inequality constraints that are valid for all k -dependent sets.

Consider any *weighting function* $\omega : V(G) \rightarrow [0, \infty)$, not everywhere zero, and let $W(\omega)$ denote the *total weight*, $\sum_{v \in V(G)} \omega(v)$, of ω . For a given k and ω , let $M_k(G, \omega)$ denote the maximum value of $\sum_{v \in V(G)} \omega(v) \cdot x_v$ over all k -dependent sets S in $V(G)$. To compute $M_k(G, \omega)$ we simply maximize the objective function $\sum_{v \in V(G)} \omega(v) \cdot x_v$ (instead of $\sum_{v \in V(G)} x_v$), using the same constraints as when computing $\beta_k(G)$ in Section 5.

We have already seen two examples of such weighting functions. One is the case where ω is the constant function $\omega(v) = 1$ (for all v), in which case $M_k(G, \omega)$ corresponds to the value of $\beta_k(G)$. The other example is

$$\omega(v) = \begin{cases} 1 & \text{if } v \in N(u), \\ \beta_k^* - k & \text{if } v = u, \\ 0 & \text{otherwise,} \end{cases}$$

for some fixed vertex u ; in this case $M_k(G, \omega)$ equals β_k^* . In general, for any ω and any k -dependent set in $V(G)$, we always have

$$\sum_{v \in V(G)} \omega(v) \cdot x_v \leq M_k(G, \omega),$$

simply by the definition of $M_k(G, \omega)$.

Lemma 2. *Consider any weighting function ω on a vertex-transitive graph G . Then an upper bound for the maximum density among k -dependent sets in G is $\rho_k(G) \leq M_k(G, \omega) / W(\omega)$.*

Proof. Let Γ denote the group of all automorphisms on G and let F denote $|\{f \in \Gamma : f(v) = v\}|$, a number which is independent of the choice of $v \in V(G)$. Consider a k -dependent set S in $V(G)$. If x_v is the characteristic vector for S and f is some automorphism, then the vector $x'_v = x_{f(v)}$ is also the characteristic vector of a k -dependent set. Therefore,

$$\sum_{v \in V(G)} \omega(v) \cdot x_{f(v)} \leq M_k(G, \omega)$$

holds for each automorphism $f \in \Gamma$. Summing these inequalities, one per automorphism f , we see that each variable x_v appears on the left-hand side with total coefficient equal to $W(\omega)F$. Thus we obtain the inequality

$$W(\omega)F|S| \leq M_k(G, \omega)|\Gamma|.$$

Using the fact [11, p. 89] that $|\Gamma| = F|V(G)|$, we obtain

$$|S| \leq \frac{M_k(G, \omega)}{W(\omega)}|V(G)|,$$

proving the claim. ■

The preceding lemma gives a very general tool for bounding the toroidal limiting density from above by means of relatively small nontoroidal kings graphs, as shown in the next result. Note that this is the only place where we explicitly consider k -dependence on a *nontoroidal* kings graph.

Lemma 3. *For any weighting function ω on $K[n_1, \dots, n_d]$, we have*

$$\rho_k^{(d)} \leq \frac{M_k(K[n_1, \dots, n_d], \omega)}{W(\omega)}.$$

Furthermore, $\inf_{\omega} [M_k(K^{(d)}[n], \omega) / W(\omega)] \rightarrow \rho_k^{(d)}$ as $n \rightarrow \infty$.

Proof. Consider any weighting function ω for $K[n_1, \dots, n_d]$. Then for all $n > n_1, n_2, \dots, n_d$, consider the weighting function ω' on $K_{\text{tor}}^{(d)}[n]$ defined by

$$\omega'(\mathbf{v}) = \begin{cases} \omega(\mathbf{v}) & \text{if } \mathbf{v} \in V(K[n_1, \dots, n_d]), \\ 0 & \text{otherwise.} \end{cases}$$

The intersection of each k -dependent subset of $V(K_{\text{tor}}^{(d)}[n])$ with $V(K[n_1, \dots, n_d])$ is k -dependent in $K[n_1, \dots, n_d]$, and any k -dependent set in $K[n_1, \dots, n_d]$ is k -dependent in $K_{\text{tor}}^{(d)}[n]$; thus $M_k(K_{\text{tor}}^{(d)}[n], \omega') = M_k(K[n_1, \dots, n_d], \omega)$. Because $K_{\text{tor}}^{(d)}[n]$ is vertex-transitive, Lemma 2 implies that

$$\rho_k^{(d)}(n) \leq \frac{M_k(K_{\text{tor}}^{(d)}[n], \omega')}{W(\omega')} = \frac{M_k(K[n_1, \dots, n_d], \omega)}{W(\omega)}$$

for all suitably large n . Letting $n \rightarrow \infty$ completes the proof of the inequality.

To prove the second statement, consider the case where $n_i = n$ for all i and ω is the constant function $\omega(\mathbf{v}) \equiv 1$. If S is a k -dependent set of maximum density in $K^{(d)}[n]$, then its characteristic vector maximizes $\sum_{\mathbf{v}} \omega(\mathbf{v})x_{\mathbf{v}}$. Viewing $K^{(d)}[n]$ as a subset of $K_{\text{tor}}^{(d)}[n+1]$, we obtain

$$\frac{|S|}{(n+1)^d} \leq \rho_k^{(d)} \leq \frac{M_k(K^{(d)}[n], \omega)}{W(\omega)} = \frac{|S|}{n^d}.$$

This implies that

$$\rho_k^{(d)} \leq \frac{M_k(K^{(d)}[n], \omega)}{W(\omega)} \leq \rho_k^{(d)} \left(\frac{n+1}{n} \right)^d,$$

so that $M_k(K^{(d)}[n], \omega) / W(\omega)$ can be made arbitrarily close to $\rho_k^{(d)}$ by choosing n sufficiently large. ■

In the introduction, we informally stated the following conjecture.

Conjecture 2. *The limiting densities for 4- and 5-dependent toroidal kings graphs in two dimensions are $\rho_4^{(2)} = 3/5$ and $\rho_5^{(2)} = 9/13$, respectively.*

Our next result supports this conjecture.

Theorem 4. *The limiting densities for 4- and 5-dependent toroidal kings graphs in two dimensions satisfy the bounds*

$$0.6 = \frac{3}{5} \leq \rho_4^{(2)} \leq 0.608956$$

and

$$0.6923 \approx \frac{9}{13} \leq \rho_5^{(2)} \leq 0.693943.$$

Proof. The lower bounds were verified in Corollary 1. We take the following general approach for finding an upper bound for $\rho_k^{(2)}$. First, choose a specific value of n (not too large), and then (carefully) choose a weighting function ω for $K[n, n]$. Next, use binary linear programming (involving n^2 binary variables) to find $M_k(K[n, n], \omega)$. This yields the upper bound

$$\rho_k^{(2)} \leq \frac{M_k(K[n, n], \omega)}{W(\omega)},$$

as given by Lemma 3.

For instance, using the weighting function ω_1 shown in Table 2 for $K[10, 10]$ with total weight $W(\omega_1) = 280$, we computed $M_4(K[10, 10], \omega_1) = 171$. Similarly, the weighting function ω_2 in Table 3 for $K[11, 11]$ with $W(\omega_2) = 2656$ gave us $M_5(K[11, 11], \omega_2) = 1844$. These calculations prove the upper bounds

$$\rho_4^{(2)} \leq \frac{171}{280} \approx 0.61071 \quad \text{and} \quad \rho_5^{(2)} \leq \frac{1844}{2656} = \frac{461}{664} \approx 0.69428.$$

The tighter bounds stated in the theorem for $\rho_4^{(2)}$ and $\rho_5^{(2)}$ were obtained by using significantly more complicated weighting functions on $K[12, 12]$ and $K[13, 13]$, respectively. These weighting functions have been posted on the web [12]. ■

Table 2. Example weighting function ω_1 for $k = 4$ in Theorem 4.

0	1	1	1	1	1	1	1	1	0
1	2	3	3	2	2	3	3	2	1
1	3	5	4	4	4	4	5	3	1
1	3	4	6	5	5	6	4	3	1
1	2	4	5	7	7	5	4	2	1
1	2	4	5	7	7	5	4	2	1
1	3	4	6	5	5	6	4	3	1
1	3	5	4	4	4	4	5	3	1
1	2	3	3	2	2	3	3	2	1
0	1	1	1	1	1	1	1	1	0

Table 3. Example weighting function ω_2 for $k = 5$ in Theorem 4.

2	4	6	6	6	6	6	6	6	4	2
4	11	15	17	20	22	20	17	15	11	4
6	15	23	27	36	40	36	27	23	15	6
6	17	27	36	45	50	45	36	27	17	6
6	20	36	45	49	49	49	45	36	20	6
6	22	40	50	49	48	49	50	40	22	6
6	20	36	45	49	49	49	45	36	20	6
6	17	27	36	45	50	45	36	27	17	6
6	15	23	27	36	40	36	27	23	15	6
4	11	15	17	20	22	20	17	15	11	4
2	4	6	6	6	6	6	6	6	4	2

In the above proof, we say that the weighting function should be chosen carefully. In fact, ω_1 and ω_2 were chosen to yield the best possible upper bounds for $K[10, 10]$ and $K[11, 11]$, respectively. We close this section with a brief explanation of how these were found.

Suppose we have a finite collection \mathcal{S} of k -dependent subsets of $V(K[n, n])$. Consider the problem of minimizing a scalar θ over all pairs (θ, ω) subject to the constraints

$$\begin{aligned} \omega(\mathbf{v}) &\geq 0, \quad \forall \mathbf{v} \in V(K[n, n]), \\ \sum_{\mathbf{v} \in V(G)} \omega(\mathbf{v}) &= 1, \\ \sum_{\mathbf{v} \in V(G)} \omega(\mathbf{v}) \cdot x_{\mathbf{v}} &\leq \theta, \quad \forall S \in \mathcal{S}, \end{aligned}$$

where we identify a set $S \in \mathcal{S}$ with its characteristic vector $(x_{\mathbf{v}} : \mathbf{v} \in V(K[n, n]))$. This optimization problem is a continuous-variable LP that can be solved in just seconds even when $|\mathcal{S}| \approx 100,000$, provided that $n < 50$. Note that if the collection \mathcal{S} contained all k -dependent subsets of $V(K[n, n])$, then the solution of this continuous LP would satisfy $\theta = M_k(K[n, n], \omega)$. Moreover, this θ would be the best possible upper bound using $K[n, n]$ under Lemma 3. However, \mathcal{S} can also have this property and be considerably smaller than the entire collection of k -dependent sets. To check if \mathcal{S} is sufficient for this purpose, solve the binary LP to maximize the ω -weight, then obtain the value $M_k(K[n, n], \omega)$ and a corresponding maximum ω -weight k -dependent set S . If the value of $M_k(K[n, n], \omega)$ equals the optimal θ from the continuous LP, then the collection \mathcal{S} is sufficient and we are done; otherwise, replace \mathcal{S} by $\mathcal{S} \cup \{S\}$ and solve the continuous LP again. This procedure is necessarily finite and guaranteed to find the best bound. If terminated early it can still find a very good bound, such as those in the proof of Theorem 4.

The linear programming techniques for k -dependence and related problems lend themselves nicely to undergraduate and Master's level research projects, provided the research supervisor can help with the details of getting LP packages to perform well. In

particular, there are many opportunities for using Lemma 2 to do further research. Any Cayley graph, for instance, is vertex-transitive, so k -dependence numbers for Cayley graphs are amenable to study in this manner.

9. OTHER TECHNIQUES: $\rho_1^{(d)}$ AND $\rho_3^{(2)}$. The binary LP technique used in Theorem 4 also improves the upper bounds on $\rho_1^{(2)}$ and $\rho_3^{(2)}$, but is not nearly as effective in these two cases. Fortunately, exact values for each can be found by other means.

The value $\rho_1^{(2)} = 1/3$ is a special case of the next result.

Theorem 5. *For any dimension d , we have $\rho_1^{(d)} = 2^{2-d}3^{-1}$.*

Proof. For any vertex $\mathbf{v} \in V(K_{\text{tor}}^{(d)}[n])$, define

$$B_{\mathbf{v}} = \{\mathbf{y} \in N(\mathbf{v}) \mid y_i - v_i \equiv 0 \text{ or } 1 \pmod{n}\}.$$

Clearly, $|B_{\mathbf{v}}| = 2^d$. Also, $B_{\mathbf{v}} \cap B_{\mathbf{u}} \neq \emptyset$ precisely when \mathbf{v} and \mathbf{u} are neighbors. Now consider any 1-dependent set $S \subseteq V(K_{\text{tor}}^{(d)}[n])$ of maximum cardinality. Suppose that $\mathbf{v}, \mathbf{u} \in S$ are neighbors for which $v_i - u_i \equiv 1 \pmod{n}$ for some i . If $\mathbf{y} \in B_{\mathbf{v}} \cap B_{\mathbf{u}}$, then $y_i = v_i$ and $|B_{\mathbf{v}} \cap B_{\mathbf{u}}| \leq 2^{d-1}$. Because S is 1-dependent, there are at most $|S|/2$ pairs $\{\mathbf{v}, \mathbf{u}\}$ for which $B_{\mathbf{v}}$ and $B_{\mathbf{u}}$ intersect. Therefore

$$|K_{\text{tor}}^{(d)}[n]| \geq \left| \bigcup_{\mathbf{v} \in S} B_{\mathbf{v}} \right| \geq |S| \cdot 2^d - \frac{|S|}{2} 2^{d-1} = |S| \cdot 2^{d-2} 3,$$

so $\rho_1^{(d)} \leq 2^{2-d}3^{-1}$. For the reverse inequality, consider $G = K_{\text{tor}}[n_1, \dots, n_d]$ with n_1 divisible by 3 and n_i even for each $i > 1$. Then the 1-dependent set

$$\{\mathbf{v} \in V(G) \mid v_1 \not\equiv 0 \pmod{3} \text{ and } v_i \equiv 0 \pmod{2}, \forall i > 1\}$$

has cardinality

$$\frac{2n_1}{3} \frac{n_2}{2} \cdots \frac{n_d}{2} = \frac{|V(G)|}{2^{d-2}3},$$

so $\rho_1^{(d)} \geq \rho_1(K_{\text{tor}}[n_1, \dots, n_d]) \geq 2^{2-d}3^{-1}$. ■

We close by deriving the exact value $\rho_3^{(2)} = 1/2$, which is surprising in that we cannot improve upon the 2-dependent density by admitting a third neighboring king. Our proof uses a “taxation” argument, such as in [4].

Theorem 6. *In $K_{\text{tor}}[m, n]$, every 3-dependent set has at most $mn/2$ vertices. Therefore, $\rho_3^{(2)} = 1/2$.*

Sketch of proof. Consider a 3-dependent set S of vertices in $K_{\text{tor}}[m, n]$. The idea of taxation used here starts with \$1 at every vertex *not* in S , and then redistributes those funds in such a way that each member of S receives at least \$1. After redistribution, the vertices in S collectively share a total of at least $|S|$ dollars, whereas the same total cannot exceed the $|V(K_{\text{tor}}[m, n]) \setminus S|$ dollars originally distributed over the vertices in the complement of S . Therefore $|V(K_{\text{tor}}[m, n])| - |S| \geq |S|$, and so $mn/2 \geq |S|$. The theorem then follows by observing that $\rho_3^{(2)} \geq \rho_2^{(2)} = 1/2$.

A taxation argument hinges on finding a suitable rule for redistributing funds. To describe our rule, we view the vertices as squares on a toroidal chessboard. For each vertex \mathbf{v} of $K_{\text{tor}}[m, n]$, let $N_{\text{side}}(\mathbf{v})$ denote the set of neighbors of \mathbf{v} that share a side with \mathbf{v} and let $N_{\text{corner}}(\mathbf{v})$ denote the set of neighbors that do not share a side with \mathbf{v} . Let $r_0(\mathbf{v})$ denote the amount of money initially available at vertex \mathbf{v} , so that $r_0(\mathbf{v}) = 1$ for $\mathbf{v} \notin S$ and $r_0(\mathbf{v}) = 0$ for $\mathbf{v} \in S$; let $t(\mathbf{v}) = 1 - r_0(\mathbf{v})$ denote the “target” amount for \mathbf{v} . We redistribute the money by the following three steps, where $r_i(\mathbf{v})$ denotes the amount at vertex \mathbf{v} immediately after step i :

1. Each \mathbf{v} distributes its “surplus” $\max\{r_0(\mathbf{v}) - t(\mathbf{v}), 0\}$ evenly among those $\mathbf{u} \in N_{\text{side}}(\mathbf{v})$ for which $r_0(\mathbf{u}) < t(\mathbf{u})$, if any.
2. Each \mathbf{v} with $r_1(\mathbf{v}) > t(\mathbf{v})$ transfers the amount $\max\{t(\mathbf{u}) - r_1(\mathbf{u}), 0\}$ to each $\mathbf{u} \in N_{\text{corner}}(\mathbf{v})$.
3. Each \mathbf{v} transfers the amount $\max\{r_2(\mathbf{v}) - t(\mathbf{v}), 0\}$ to each $\mathbf{u} \in N_{\text{side}}(\mathbf{v})$ for which $r_2(\mathbf{u}) < t(\mathbf{u})$.

A case-by-case examination of the neighborhood possibilities for members of the 3-dependent set S verifies that $r_i(\mathbf{v}) \geq 0$ and $r_3(\mathbf{v}) \geq t(\mathbf{v})$ for all \mathbf{v} . ■

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Mathematics Is . . .

“Pure mathematics is a sucker’s game. It lures the curious and confident with its seeming simplicity only to make them look like fools.”

Sharon Begley, *New answers for an old question,*
Newsweek, July 5, 1993, p. 52.

—Submitted by Carl C. Gaither, Killeen, TX