ACADÉMIE ROUMAINE

REVUE ROUMAINE DE MATHÉMATIQUES PURES ET APPLIQUÉES

(ROMANIAN JOURNAL OF PURE AND APPLIED MATHEMATICS)

TOME XXXVII, N° 8 1992

TIRAGE À PART

EDITURA ACADEMIEI ROMÂNE

JORDAN DECOMPOSITION AND FACTORIZATION FOR NONNEGATIVE OPERATOR VALUED FUNCTIONS

EUGEN-JULIEN IONAŞCU

§1. INTRODUCTION

In the present paper we give new proofs to some results in [4], where St. Frunză uses techniques from the decomposable operator theory. Our approach is different and is based on the well-known theorems of factorization for nonnegative operator valued functions (see [7], [8]).

The proofs have an algebraical character with a few exceptions. Therefore, the facts can be presented in an arbitrary C^*-algebra and we will mention that when it will be the case.

To be more precise, let $\mathscr{H}$ be a complex Hilbert space and $L(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. The connection between these two types of decomposition, will be made with the aid of the map

$L(\mathscr{H}) \ni T \to \Phi_T$, where $\Phi_T : \mathbb{R} \to L(\mathscr{H})$ is defined by $\Phi_T(t) = \exp(-it\ T^*) \exp(it\ T)$ for every $t \in \mathbb{R}$.

This map is a very well-known one (see [7], [8]). It is clear that $\Phi_T(t) \geqslant 0$ for all real t, and if $T \in L(\mathscr{H})$ have in addition some properties, these ones will be reflected by Φ_T in some way and reciprocally. In our case Φ_T satisfy the following theorem from [7] (th. 3.3):

THEOREM A. *Let* $P(t) = \sum\limits_{j=0}^{2n} P_j t^j$ *be a polynomial whose coefficients are operators on* $\mathscr{H}$ *and which is nonnegative on* $\mathbb{R}$. *Then* $P = Q^*Q$ *where* Q *is an outer function on* $\mathbb{R}$ *of the form* $Q(t) = \sum\limits_{j=0}^{n} Q_j t^j$ *for some operators* $Q_0, Q_1, \ldots, Q_n$ on.

This must happen if T has a Jordan decomposition, $T = N + Q$, where $N \in L(\mathscr{H})$ is a normal operator and Q a nilpotent operator of some order k which commutes with N.

In the case $k = 2$, we give an independent elementary proof and accurate formulas for N and Q as functions of T and T^*. Unfortunately, in the general case such formulas seem to be very complicated.

§2. PRELIMINARIES

The commutator $C(T, S)$ of two operators $T, S \in L(\mathscr{H})$ is the operator defined on $L(\mathscr{H})$ by $C(T, S)X = TX - XS$ for all $X \in L(\mathscr{H})$. Following [2] we define the relation n_k on $L(\mathscr{H})$ by

(1) $$Tn_k S \text{ if } C^k(T, S)(I) = 0 \ (k \in N^*)$$

and I being the identity operator on $\mathscr{H}$.

REV. ROUMAINE MATH. PURES APPL., 37(1992), 8, 691–699

An operator $T \in L(\mathscr{H})$ will be called Jordan operator of order if it has a decomposition $T = A + N$, where A is selfadjoint and $\in L(\mathscr{H})$ a nilpotent of order k, commuting with A. It is easy to see th

$$C^k(T, S)(I) := (T - S)^{[k]} = T^k - \binom{k}{1} T^{k-1}S + \dots + \tag{2}$$

$$+ (-1)^k S^k \ (k \in N^*)$$

and

$$\exp(zT)\ \exp(-zS) = I + \sum_{j=1}^{\infty} (T - S)^{[j]} \frac{z^j}{j!} \tag{3}$$

for all $z \in \mathbb{C}$. If $T\, n_k\, S$ for some k, then the above entire function mu be polynomial. As we have already seen, Φ_T and Φ_{T^*} are nonnegative oper tor valued functions on R, which are given by:

$$\Phi_T(t) = \exp(-it\ \ T^*)\ \exp\ (it\ \ T) \tag{4}$$

$$\Phi_{T^*}(t) = \exp\ (-it\ \ T)\ \exp\ (it\ \ T^*) \tag{5}$$

If $T^* n_k\, T$ for some $k \in N^*$, it is not compulsory that $T n_k T^*$, as show in [5], by considering in the place of T, the multiplication with t variable on a weighted Sobolev space on $\mathbb{R}$, restricted to an invaria subspace.

If Φ_T is polynomial, since Φ_T is nonnegative, it must have ev degree. Indeed if $\Phi_T(t) \doteq \sum\limits_{j=1}^{2n+1} P_j t^j$ then $P_{2n+1} \doteq \lim\limits_{t\to\infty} \dfrac{\Phi_T(t)}{t^{2n+1}}$ implies P_{2n+1} $\geqslant 0$ and also $P_{2n+1} = \lim\limits_{t\to-\infty} \Phi_T(t)/t^{2n+1} \leqslant 0$, hence $P_{2n+1} = 0$. By definition $T \in L(\mathscr{H})$ are quasinilpotent equivalent and we write $S \overset{q}{\sim} T$ if $\lim\limits_n \|(S - T)^{[n]}\|^{1/n} = \lim\limits_n \|(T - S)^{[n]}\|^{1/n} = 0$.

§3. THE GOAL OF THIS SECTION IS TO PROVE THE FOLLOWING THEOREM WHICH CHARACTERIZES THE JORDAN OPERATORS IN TERMS OF T AND

THEOREM 3.1. *Let $T \in L(\mathscr{H})$ and $k \in N^*$. The following two co ditions are equivalent:*

(i) *T is a Jordan operator of order k*

(ii) *$T^* n_{2k-1} T$ and $T n_{2k-1} T^*$* (see (1)).

If (i) holds, $T = A + N$ where $A = A^*$ and N is nilpotent order k commuting with A. Then we have

$$C^{2k-1}(T^*, T)(I) = C^{2k-1}(N^*, N)(I) = 0$$

corresponding to (2) and the fact that $N^k = N^{*k} = 0$.

Hence (ii) follows. Obviously $k = 1$ is a trivial case. For the converse we give first the proof of the case $k = 2$, since as we say, the treatment is elementary and we get exact formulas for A and N as functions of T and T^*.

According to the observation made in §2, (ii) implies that there are equalities:

$$\Phi_T(t) = I + A_1 t + A_2 t^2 \text{ and } \Phi_{T^*}(t) = I + B_1 t + B_2 t^2 \text{ for all } t \in \mathbb{R}$$

where

$$A_1 = -B_1 = i(T^* - T), \quad A_2 = -\frac{1}{2}(T^2 - 2TT^* + T^{*2}),$$

$$B_2 = -\frac{1}{2}(T^{*2} - 2T^*T + T^2)$$

From the fact that $\Phi_T(t)\Phi_{T^*}(t) = \Phi_{T^*}(t)\Phi_T(t) = I$ for every real t, it follows

$$A_2B_2 = B_2A_2 = 0, \; A_1B_2 + A_2B_1 = 0, \; A_2 + A_1B_1 + B_2 = 0, \tag{6}$$

so that

$$A_2 + B_2 = A_1^2 = B_1^2. \tag{7}$$

Let $C = iB_1(A_2 - B_2)$, which is selfadjoint since from (6)

$$C^* = -i(A_2 - B_2)B_1 = iA_1B_2 + iB_1A_2 = C.$$

Then there exists $R = C^{1/3}$, the selfadjoint cubic root, obtained by the continuous functional calculus for normal operators.

Let us show that the operator

$$N = \frac{1}{2}(iA_1 + R) \tag{8}$$

is the desired nilpotent of order 2. First of all, let us check the next two properties of R

$$R^2 = A_1^2 \text{ and } RA_1 + A_1R = 0. \tag{9}$$

Indeed we may write the sequence

$$R^2 = C^{2/3} = (C^2)^{1/3} = [B_1(A_2 - B_2)(A_2 - B_2)B_1]^{1/3} =$$

$$= [B_1(A_2^2 + B_2^2)B_1]^{1/3} = [B_1(A_2 + B_2)^2 B_1]^{1/3} =$$

$$= [B_1^6]^{1/3} = A_1^2.$$

Certainly we have used again relations (6).

For the second equality from (9), let us take a sequence of polynomials (P_n) with $P_n(0) = 0$ uniformly convergent on the spectrum of C, to the map $t \to t^{1/3}$.

Hence, it is easy to see that $P_n(C)B_1 + B_1P_n(C) = 0$ for all nonnegative integers n. Then, after passing to the limit, we get $RA_1 + A_1R = 0$.

Therefore, equalities (9) imply that

$$N^2 = 1/4[R^2 - A_1^2 + i(RA_1 + A_1R] = 0.$$

To finish the proof, it is enough to show that the operator $A = T - N$ is selfadjoint and commutes with N. Let us notice that N can be written in the form

$$N = 1/2 \ \{T - T^* - [(T - T^*)(TT^* - T^*T)]^{1/3}\} \tag{10}$$

and then A has the form

$$A = 1/2 \ \{T + T^* + [(T - T^*)(TT^* - T^*T)]^{1/3}\} \tag{11}$$

For commutativity we use the following lemma which has, in some, way, an independent character

LEMMA 3.2. *Let* $\mathscr{A} \subset L(\mathscr{H})$ *be a* C^**-algebra,* $\mathscr{N}(\mathscr{A}) = \{N | N \in \mathscr{A}$ $N^2 = 0\}$ *and the maps* σ_+, σ_- *defined by*
(12) $\sigma_\pm(N) = 1/2[N - N^* \pm (N^*N)^{1/2} \mp (NN^*)^{1/2}]$ *for all* $N \in \mathscr{N}(A)$. *Then we have*

(i) $$\sigma_\pm(\mathscr{N}(\mathscr{A})) \subset \mathscr{N}(\mathscr{A})$$

(ii) $$\sigma_+ \circ \sigma_- = \sigma_- \circ \sigma_+ = \mathrm{id}$$

(iii) $N \in \mathscr{N}(A)$ *commutes with a selfadjoint operator* $A \in L(\mathscr{H})$ *if and only if* A *commutes with* $N - N^*$ *and* NN^*(or N^*N).

Proof. Let us denote by $R_0 = (N^*N)^{1/2} - (NN^*)^{1/2}$. Then $4\sigma_+^2(N) = (N - N^*)^2 + R_0^2 + (N - N^*)R_0 + R_0(N - N^*) = 0$ if we show that $R_0^2 + (N - N^*)^2 = 0$ and $(N - N^*)R_0 + R_0(N - N^*) = 0$. We choose analogously a sequence of polynomials $(Q_n)_{n>0}$, with $Q_n(0) = 0$ uniformly convergent on the interval $[0, \|N\|^2]$ to the function $t \to t^{1/2}$.

Hence, for every nonnegative integer n, it is not difficult to see that we have the equalities:

$$Q_n(N^*N)Q_n(NN^*) = 0, \quad NQ_n(N^*N) = Q_n(NN^*)N$$

and

$$NQ_n(NN^*) = Q_n(N^*N)N = 0.$$

Then, after passing to the limit, it follows that

$$(N^*N)^{1/2}(NN^*)^{1/2} = 0, \quad N(N^*N)^{1/2} = (NN^*)^{1/2}\ N \text{ and } N(NN^*)^{1/2} = 0$$

Finally, these relations imply the desired ones:

$$(13) \qquad R_0^2 = NN^* + N^*N = -(N - N^*)^2, \quad R_0(N - N^*) + (N - N^*)R_0 = 0$$

Analogously $\sigma_-(N)^2 = 0$. This shows (i)
For the point (ii), let us compute $\sigma_-(\sigma_+(N))$:

$$\sigma_-(\sigma_+(N)) = \frac{1}{2}(\sigma_+(N) - \sigma_+(N)^* - [\sigma_+(N)^*\sigma_+(N)]^{\frac{1}{2}} + [\sigma_+(N)\sigma_+(N)^*]^{\frac{1}{2}}) = \frac{1}{2}(N - N^* - [\sigma_+^*\sigma_+]^{\frac{1}{2}} + [\sigma_+\sigma_+^*]^{1/2}).$$

This time, from (13) we have

$$\sigma_+^*\sigma_+ = \frac{1}{4}[R_0^2 - (N - N^*)^2 - (N - N^*)R_0 + R_0(N - N^*)] =$$

$$= \frac{1}{2}[NN^* + N^*N - (N - N^*)R_0] =$$

$$= \frac{1}{4}[|N + N^*| - (N + N^*)]^2$$

where $|N + N^*| = (NN^* + N^*N)^{1/2}$. Analogously

$$\sigma_+\sigma_+^* = \frac{1}{4}[|N + N^*| + N + N^*]^2$$

and by the uniquenes of the nonnegative square root, we infer that

$$(\sigma_+^*\sigma_+)^{\frac{1}{2}} = \frac{1}{2}(|N + N^*| - N - N^*), \quad (\sigma_+\sigma_+^*)^{\frac{1}{2}} = \frac{1}{2}(|N + N^*| + N + N^*).$$

Therefore $\sigma_-(\sigma_+(N)) = \frac{1}{2}(N - N^* + N + N^*) = N$ which proves (ii).

The equivalence from (iii) is easy to prove, in one way. Let us suppose that the selfadjoint operator A commutes with $N - N^*$ and N^*N. Then A commute with NN^*, since $NN^* + N^*N + (N - N^*)^2 = 0$.

Hence, A commutes with $(N^*N)^{1/2}$, $(NN^*)^{1/2}$ and then with $\sigma_+(N)$ and $\sigma_+^*(N)$. Finally, A commutes with $\sigma_-(\sigma_+(N)) = N$ as required.

Returning to the proof of Theorem 3.1, on the account of the above lemma, we must show that A commutes with $N - N^* = T - T^*$ and NN^*.

Using (11), the commutation with $T - T^*$ becomes $2(TT^* - T^*T) = i(RA_1 - A_1R)$ which, by equalities (9), this is the same with:

$$iRA_1 = T^*T - TT^*. \tag{14}$$

From this we obtain:

$$-iRA_1RA_1RA_1 = iR^3A_1 = (T^*T - TT^*)^3$$

But $R^3 = iB_1(A_2 - B_2)$ and then

$$iR^3A_1^3 = -B_1(A_2 - B_2)A_1^3 = (B_2 - A_2)A_1^4 =$$

$$= (B_2 - A_2)(B_2 + A_2)^2 = (B_2 - A_2)^3$$

and here again by (9), $B_2 - A_2 = T^*T - TT^*$.
Hence, (14) holds.

To prove the commutation with NN^* we use the identities fulfilled by T:

$$C^3(T^*, T)(I) = C^3(T, T^*)(I) = 0$$

More precisely we have

$$C^3(T, T^*)(I) = T^2(T - T^*) - 2T(T - T^*)T^* + (T - T^*)T^{*2} = 0$$

and replacing $T = A + N$ and using the fact already proved, that A commute with $N - N^*$, we obtain the desired equality: $ANN^* = NN^*A$.

Let observe that the formulas (10) and (11) which give the operators A and N as functions of T and T^*, are specific to one case of non-commuting functional calculus.

Now, we consider the general case of Theorem 3.1. From (ii), we get that the maps Φ_T and Φ_{T^*} are polynomials of degree at most $2k-2$.

Hence, as we said, we apply Theorem A of factorization from [7] due to Rosenblum and Rovnyak, which generalizes the analogous classical result of Féjer and Riesz.

Moreover, we need the following variant of Theorem 2.4 from [7], concerning the uniqueness of such factorization.

THEOREM B. *Let G_1 and G_2 be two operator valued outer functions on* $\mathbb{R}$. *(in the sense of* [7]).

(i) *We have* $G_1^*G_1 = G_2^*G_2$ *a.e. if and only if* $G_2 = UG_1$ $G_1 = U^*G_2$ *a.e. where* U *is a (constant) partial isometric operator on* $\mathscr{H}$.

(ii) *If* $G_1^*G_1 = G_2^*G_2$, *for continuous* G_1, G_2, *and* $G_1(t_0) = G_2(t_0)$ *for some fixed* $t_0 \in \mathbb{R}$, *then* $G_1 = G_2$.

The proof of this variant is obvious, via the original result from [7].

By these theorems, there are unique polynomials u and v whose coefficients are operators on $\mathscr{H}$, such that

$$\Phi_T(t) = U^*(t)\ u(t),\ \Phi_{T^*}(t) = v(t)\ v^*(t) \text{ for all } t \in \mathbb{R}. \tag{15}$$

$$u(0) = v(0) = I,\ u(t) = \sum_{j=0}^{k-1} U_j t^j,\ v(t) = \sum_{j=0}^{k-1} V_j t^j \tag{16}$$

and u, v are outer functions on $\mathbb{R}$ (in the sense of [7]).

Let us show first that $u(t)\ v(t) = v(t)\ u(t) = I$ for every real t.

Since $\Phi_T(t)\ \ \Phi_{T^*}(t) = \Phi_{T^*}(t)\ \ \Phi_T(t) = I$, we have:

$$u^*(t)\ u(t)\ v(t)\ v^*(t) = v(t)\ v^*(t)\ u^*(t)\ u(t) = I\ \ (t \in \mathbb{R})$$

But for small $t > 0$, $u(t)$ and $v(t)$ are invertible operators. That implies the next equalities:

$$(u(t)\ v(t))^* u(t)\ v(t) = u(t)\ v(t)\ (u(t)\ v(t))^* = I \tag{17}$$

We know that $u(t)\ v(t) = \sum_{j=0}^{s} W_j t^j$ for some integer $s > 0$ and $W_j \in L(\mathscr{H})$. From (17) we see that $W_s^* W_s = 0$ if $s > 0$. This implies the fact that $W_j = 0$ for $j > 0$ and then

$$u(t)\ v(t) = v(t)\ u(t) = I$$

in fact for all $t \in \mathbb{R}$.

Now, for t and s in R we can write, by (15)

$$u^*(t + s)u(t + s) = \exp\ (-\text{is}\ \ T^*)u^*(t)\ u(t)\ \exp\ (\text{is}\ \ T) \text{ or}$$

$$u^*(t + s)\, u(t + s) = [u(s)\exp(-\text{is}\ \ T)\ \ u(t)\ \ \exp(\text{is}\ \ T)]^*\ u(s)\cdot \tag{18}$$

$$\cdot\exp(-\text{is}\ \ T)u(t)\ \ \exp(\text{is}\ \ T).$$

Let us observe that the map $t \to u(s)\ \exp\ (-\text{is}\ \ T)u(t)\ \exp\ (\text{is}\ \ T)$, for very fixed $s \in \mathbb{R}$, is the nontangential limit of some outer function on the half-plane $y > 0$ to $\mathbb{R}$ (in sense of [7]).

This happenes since if G is outer on $\mathbb{R}$ and $X \in L(\mathscr{H})$ then it is also true for the maps $t \to XG(t)$ and $t \to G(t)X$. Therefore (18) and Theorem **B** (ii) give us the following

$$u(t + s) = u(s)\ \exp\ (-\text{is}\ \ T)\ u(t)\ \exp(\text{is}\ \ T). \tag{19}$$

If we differentiate (19) with respect to s, and evaluate it at $s = 0$ we get

$$u'(t) = u'(0)u(t) + i[u(t),\ T] \tag{20}$$

where $[A,\ B] = AB - BA$ for $A,\ B \in L(\mathscr{H})$.

From (15), an identication of coefficients gives $U_1^* + U_1 = iT - iT^*$ or $T + iU_1 = (T + iU_1)^*$. Then we can define the selfadjoint operator $A = T + iU_1$ and $N = -iU_1$.

Solving equation (20) with these new notation we have

$$u(t) = \exp(-\mathrm{it}\ A) \exp\ (\mathrm{it}\ T). \tag{21}$$

Then if we invert this, we get

$$v(t) = \exp\ (-\mathrm{it}\ T) \exp(\mathrm{it}\ A) \tag{22}$$

Now, let us write that u and v are polynomials of degree at most $k - 1$:

$$C^k(A,\ T)(I) = C^k(T,\ A)(I) = 0 \tag{23}$$

For $k = 2$ it is easy to see that (23) implies the commutation of A and T. This is true in general and we sketch the proof for $k = 3$, since it goes analogous for every k. Therefore (23) becomes

$$A^3 - 3A^2T + 3AT^2 - T^3 = T^3 - 3T^2A + 3TA^2 - A^3 = 0 \tag{24}$$

Since A is selfadjoint, there exists a spectral measure E concentrated on $\mathbb{R}$ (see [6]) which gives a spectral resolution of the identity $E(t) = E((-\infty,\ t])$ and we have the representation

$$A = \int_{\mathbb{R}} t dE(t)$$

Let $\sigma = [a,\ b]$ and $\omega = [c,\ d]$ two closed and disjoint intervals and γ_1, γ_2 two closed Jordan curves, which surround σ and ω respectively and have no intersection. Then we get

$$E(\omega) = \frac{1}{2\pi i}\int_{\gamma_2} f(\xi) d\xi$$

where $f(\xi) = \displaystyle\int_{\mathbb{R}} \frac{1}{\xi - t}\, \chi_\omega(t)\ dE(t)$ is analytic on $\mathbb{C} \setminus \omega$.

Now, we use some ideas from [2] where we can find a generalization of the present result:

Proposition 3.3. *Let $T \in L(\mathscr{H})$ and $A \in L(\mathscr{H})$ selfadjoint such that $T \overset{\cdot}{\sim} A$. Then T and A commute.*

By computations from (24) the function $\tilde{f}(\xi)=f(\xi)-\frac{T-A}{1!}f'(\xi)+\frac{(T-A)^{[2]}}{2!}f''(\xi)$ satisfies the equality $(\xi-T)\tilde{f}(\xi)=E(\omega)$ for $\xi \in C\setminus\omega$ and then the map $g(\xi)=T\tilde{f}(\xi)-\frac{A-T}{1!}T\tilde{f}'(\xi)+\frac{(A-T)^{[2]}}{2!}T\tilde{f}''(\xi)$ satisfies the equality $(\xi-A)g(\xi)=TE(\omega)$ for all $\xi \in \mathbb{C}\setminus\omega$. Then we have

$$E(\sigma)TE(\omega)=\frac{1}{2\pi i}\int_{\gamma_1}(\xi-A)^{-1}E(\sigma)TE(\omega)d\xi=\frac{1}{2\pi i}\int_{\gamma_1}E(\sigma)g(\xi)d\xi=0 \quad (\sigma\subset C\setminus\omega).$$

Since ω is arbitrary in $R\setminus\sigma$, this implies that $E(\sigma)T(I-E(\sigma))=0$. The same is true for T^* and then $TE(\sigma)=E(\sigma)T$ or equivalently $TA=AT$. The proof of Proposition 3.3 can be made in the same fashion. Returning to the proof of Theorem 3.1, since A and T commute, (23) says that $N=T-A$ is nilpotent of order k and that finishes the implication (ii) $\Rightarrow$ (i).

Lemma 3.2 is in fact inspired from the work of Helton [5].

Received November 25, 1990

Institute of Mathematics
of the Romanian Academy
Bucharest, Romania

REFERENCES

1. J. Agler, *Sub-Jordan operators: Bishop's theorem, spectral inclusion and spectral sets*. J. Operator Theory, **7** (1982), 373–395.
2. I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*. Gordon and Breach, New York, 1968.
3. N. Dunford and J. T. Schwartz, *Linear Operators*, III, *Spectral Operators*. Interscience Publishers, New York, 1971.
4. Şt. Frunză, *Jordan operators on Hilbert spaces*. J. Operator Theory, **13** (1987), 201–212.
5. J. W. Helton, *Infinite dimensional Jordan operators and Sturm-Liouville conjugate point theory*. Trans. Amer. Math. Soc., **170** (1972), 305–331.
6. Martin Schechter, *Principles of Functional Analysis*. Acad. Press, New York and London.
7. M. Rosemblum and J. Rovnyak, *The factorization problem for nonnegative operator valued functions*. Bull. Amer. Math. Soc., **77**, *3* (1971).
8. M. Rosemblum and J. Rovnyak, *Hardy Classes and Operator Theory*. New York, Oxford Univ. Press, 1985.