EFFECTIVE ASYMPTOTICS FOR SOME NONLINEAR RECURRENCES AND ALMOST DOUBLY-EXPONENTIAL SEQUENCES

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We develop a technique to compute asymptotic expansions for recurrent sequences of the form $a_{n+1} = f(a_n)$, where $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers α, β, a, and b satisfying $0 < a,\ 1 < \alpha < \beta$. We prove a result which summarizes the present stage of our investigation, generalizing the expansions in [2]. One can apply our technique, for instance, to obtain the formula: $a_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10}\frac{\ln n}{n\sqrt{n}} + \frac{9\sqrt{3}}{50}\frac{\ln n}{n^2\sqrt{n}} + o\left(\frac{\ln n}{n^{5/2}}\right)$, where $a_{n+1} = \sin(a_n)$, $a_1 \in \mathbb{R}$. Moreover, we consider the recurrences $a_{n+1} = a_n^2 + g_n$, and we prove that under some technical assumptions, a_n is almost doubly-exponential, namely $a_n = \lfloor k^{2^n} \rfloor$, $a_n = \lfloor k^{2^n} \rfloor + 1$, $a_n = \lfloor k^{2^n} - \frac{1}{2} \rfloor$, or $a_n = \lfloor k^{2^n} + \frac{5}{2} \rfloor$ for some real number k. generalizing a result of Aho and Sloane [1].

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1. Introduction

Obtaining an *exact* formula for the terms of a sequence given by a recurrence may not, in general, be possible. It is the intent of this paper to investigate and give asymptotics for sequences given by recurrences of the form $a_{n+1} = f(a_n)$, where $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers α, β, a, and b satisfying $0 < a,\ 1 < \alpha < \beta$. We also consider the same recurrence where $f(x) = x - x^2$ and give more detailed asymptotics. Moreover, we prove a few results concerning almost doubly-exponential sequences $a_{n+1} = a_n^2 + g_n$, where $-a_n + 1 < g_n < 2a_n$, generalizing a result of Aho and Sloane [**1**]. For standard notations consult [**3**], or any other book on differential and integral calculus.

2. Asymptotics of Nonlinear Recurrences

The first part of the next lemma is known as Cesàro's lemma, and the second part is just a small variation of the first. For completeness, we include a proof of the second part of this lemma.

Lemma 1 (Cesàro). *Let $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ two sequences of real numbers satisfying one of the following conditions:*
(i) $\{v_n\}_{n\in\mathbb{N}}$ is eventually a strictly increasing sequence converging to infinity, or
(ii) $\{v_n\}_{n\in\mathbb{N}}$ is eventually a strictly decreasing sequence converging to zero, and u_n converges to zero.
If the limit of the sequence $\frac{u_{n+1}-u_n}{v_{n+1}-v_n}$ exists, then the limit of the sequence $\frac{u_n}{v_n}$ exists, and we have the equality

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{u_{n+1}-u_n}{v_{n+1}-v_n}. \tag{1}$$

Proof. Suppose we are given an $\epsilon > 0$, and by our hypothesis, for some integer n_0 and some real number l we have

$$\left| \frac{u_{n+1}-u_n}{v_{n+1}-v_n} - l \right| < \epsilon, \quad n \ge n_0.$$

Using (ii) the above inequality can be equivalently written in the form

$$-\epsilon(v_n - v_{n+1}) < u_n - u_{n+1} - l(v_n - v_{n+1}) < \epsilon(v_n - v_{n+1}), \quad n \ge n_0.$$

Adding up these inequalities from $n \geq n_0$ to some larger integer $m > n \geq n_0$, we get

$$-\epsilon(v_n - v_{m+1}) < u_n - u_{m+1} - l(v_n - v_{m+1}) < \epsilon(v_n - v_{m+1}), \quad m > n \geq n_0.$$

Letting m go to infinity in the above inequality and taking into account that $u_m \to 0$ and $v_m \to 0$, we obtain

$$-\epsilon v_n \leq u_n - l v_n \leq \epsilon v_n, \quad n \geq n_0,$$

which gives finally, after dividing by v_n, the conclusion of our lemma. □

Theorem 2. *Suppose f is a real-valued continuous function defined on the interval $I = (0, \delta)$ (for some δ), which has the form $f(x) = x - ax^\alpha + bx^\beta + o(x^\beta)$ as $x \to 0$, for some real numbers α, β, a, and b satisfying $0 < a$, $1 < \alpha < \beta$. Then, for a_0 sufficiently small, the orbit sequence $a_n = f(a_{n-1})$, satisfies one of the following:*
(i) if $\beta = 2\alpha - 1$, then

$$a_n = \frac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}} \left(\frac{1}{n}\right)^{1/(\alpha-1)} + \frac{b - \frac{a^2\alpha}{2}}{[a(\alpha-1)]^{\frac{2\alpha-1}{\alpha-1}}} \frac{\ln n}{n^{\alpha/(\alpha-1)}} + o\left(\frac{\ln n}{n^{\alpha/(\alpha-1)}}\right),$$

(ii) if $\beta > 2\alpha - 1$, then

$$a_n = \frac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}} \left(\frac{1}{n}\right)^{1/(\alpha-1)} - \frac{\frac{a^2\alpha}{2}}{[a(\alpha-1)]^{\frac{2\alpha-1}{\alpha-1}}} \frac{\ln n}{n^{\alpha/(\alpha-1)}} + o\left(\frac{\ln n}{n^{\alpha/(\alpha-1)}}\right).$$

(iii) if $\beta < 2\alpha - 1$ and $b \neq 0$, then

$$a_n = \frac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}} \left(\frac{1}{n}\right)^{1/(\alpha-1)} + \frac{b\,[a(\alpha-1)]^{\frac{\alpha-\beta-1}{\alpha-1}}}{a(2\alpha-1-\beta)} \left(\frac{1}{n}\right)^{\frac{\beta-1}{\alpha-1}} + o\left(\left(\frac{1}{n}\right)^{\frac{\beta-1}{\alpha-1}}\right).$$

Proof. We give the idea of the proof only in the case (i). Since $f(x) < x$, for x in a small neighborhood of zero, the sequence a_n is decreasing to zero if we assume also that a_0 is positive. Then we apply Cesáro's lemma for the sequences $u_n = \frac{1}{a_n^{\alpha-1}}$, and $v_n = n$:

$$\lim_n \frac{1}{n a_n^{\alpha-1}} = \lim_n \left(\frac{1}{a_{n+1}^{\alpha-1}} - \frac{1}{a_n^{\alpha-1}}\right) = \lim_n \left(\frac{1}{f(a_n)^{\alpha-1}} - \frac{1}{a_n^{\alpha-1}}\right).$$

Using the well-known formula from calculus $\lim_{x\to 0} \dfrac{1-(1-x)^\gamma}{x} = \gamma$, we obtain

$$\lim_n \frac{1}{n a_n^{\alpha-1}} = \lim_n \frac{1}{a_n^{\alpha-1}} \frac{\left(1 - \left(1 - a a_n^{\alpha-1} + b a_n^{\beta-1} + o(a_n^{\beta-1})\right)^{\alpha-1}\right)}{\left(1 - a a_n^{\alpha-1} + b a_n^{\beta-1} + o(a_n^{\beta-1})\right)^{\alpha-1}} =$$

$$\lim_n \frac{1 - \left(1 - a a_n^{\alpha-1} + b a_n^{\beta-1} + o(a_n^{\beta-1})\right)^{\alpha-1}}{a a_n^{\alpha-1} - b a_n^{\beta-1} - o(a_n^{\beta-1})} \frac{a a_n^{\alpha-1} - b a_n^{\beta-1} - o(a_n^{\beta-1})}{a_n^{\alpha-1}} = (\alpha-1)a.$$

Equivalently, this means that $a_n = \dfrac{1}{[a(\alpha-1)]^{\frac{1}{\alpha-1}}} \left(\dfrac{1}{n}\right)^{1/(\alpha-1)} + o\left(\left(\dfrac{1}{n}\right)^{1/(\alpha-1)}\right)$, which is the first approximation in the statements $(i) - (iii)$. Now let us assume that $\beta = 2\alpha - 1$. To simplify the computations we will denote $c = a(\alpha-1)$, and $y_n = a a_n^{\alpha-1} - b a_n^{\beta-1} - o(a_n^{\beta-1})$, which under the above assumption becomes $y_n = a a_n^{\alpha-1} - b a_n^{2(\alpha-1)} - o(a_n^{2(\alpha-1)})$. We want to apply Cesàro's lemma again for $u_n = cn - \frac{1}{a_n^{\alpha-1}}$ and $v_n = \ln\, n$:

$$\lim_n \frac{cn - \frac{1}{a_n^{\alpha-1}}}{\ln n} = \lim_n \frac{c - \frac{1}{a_{n+1}^{\alpha-1}} + \frac{1}{a_n^{\alpha-1}}}{\ln(1+\frac{1}{n})} =$$
$$\lim_n\ n\frac{(1-y_n)^{\alpha-1} + ca_n^{\alpha-1}(1-y_n)^{\alpha-1} - 1}{a_n^{\alpha-1}(1-y_n)^{\alpha-1}} =$$
$$c\lim_n\ n^2y_n^2\frac{(1-y_n)^{\alpha-1} - 1 + (\alpha-1)y_n}{y_n^2} + n^2\left(ca_n^{\alpha-1}(1-y_n)^{\alpha-1} - (\alpha-1)y_n\right).$$

Taking into account that $\lim_{n\to\infty} ny_n = \frac{a}{c}$ and $\lim_{y\to 0}\ \frac{(1-y)^\gamma - 1 + \gamma y}{y^2} = \frac{\gamma(\gamma-1)}{2}$, we may continue the above computation as follows:

$$\lim_n \frac{cn - \frac{1}{a_n^{\alpha-1}}}{\ln n} = \frac{a(\alpha-2)}{2} + c\lim_n(\alpha-1)n^2[aa_n^{\alpha-1}(1-y_n)^{\alpha-1}$$
$$-aa_n^{\alpha-1} + ba_n^{2(\alpha-1)} + o(a_n^{2(\alpha-1)})] = \frac{a(\alpha-2)}{2} + \frac{b}{a}$$
$$+a(\alpha-1)\lim_n\ n^2a_n^{\alpha-1}\left((1-y_n)^{\alpha-1} - 1\right) = \frac{b - \frac{a^2\alpha}{2}}{a}$$

This finally says that

$$\lim_n \frac{\left[(cn)^{1/(\alpha-1)}a_n - 1\right]n}{\ln n} = \frac{b - \frac{a^2\alpha}{2}}{c^2},$$

from which (i) can be easily derived. The rest of the cases are treated similarly. □

In Odlyzko's excellent paper [**6**], a few methods are studied for approximating nonlinear recurrences by linear ones. If $f(x) = x - x^2$, the following method for determining an approximation of a_n is presented. Let $x_n = 1/a_n$. By iteration we obtain

$$x_n = x_{n-1} + 1 + \frac{a_{n-1}}{1-a_{n-1}} = \cdots = \frac{1}{a_0} + n + \sum_{j=0}^{n-1}\frac{a_j}{1-a_j}.$$

We get that

$$n \le x_n \le n + O(\log n),$$

therefore $x_n = n + \log n + o(\log n)$. In our next theorem, we push further the technique (by a somewhat similar method). We would like to mention that the function of which orbit is studied here constitutes an important case of an one-dimensional dynamical system (see Theorem 10.1, Chap. II of [6]).

Theorem 3. *Assume $a_{n+1} = f(a_n)$, where $f(x) = x - x^2$. For each $a_1 \in I = (0,1)$, the function g defined by*

$$g(a_1) = \lim_{n\to\infty}\left(\frac{1}{a_n} - n - \ln n\right), \tag{2}$$

has the properties:
(i) g is continuously differentiable on I, and for all $x \in I$ we have
$$g(x) = g(1-x),\ \text{and}\ g(f(x)) = g(x) + 1;$$
(ii) g is strictly decreasing on $(0,1/2)$, strictly increasing on $(1/2,1)$, and its minimum value $g(1/2)$ is a positive number;
(iii) the measure $d\xi(x) = g'(x)dx$ is invariant under the action of f on $(0,1/2)$, i.e., for any measurable subset A of $(0,1/2)$ we have $\xi(A) = \xi(f(A))$;
(iv) if we denote $G_k(a_1) = \sum_{n\ge 1}^{\infty}\left(\frac{a_n}{1-a_n}\right)^k$, $k \ge 2$, then for $x \in (0,1/2)$

$$g(x) = \ln\left(C + \int_x^{1/2}\frac{1}{t}\exp\left(\frac{1}{t} - 1 - \sum_{k=2}^{\infty}\frac{1}{k}G_k(t)\right)dt\right), \tag{3}$$

where $C = \exp(g(1/2))$ *is a constant approximately equal to* $2.15768...$.
(v) *the following expansions hold:*

$$a_n = \frac{1}{n} - \frac{\ln n}{n^2} - \frac{g(a_0)}{n^2} + \frac{(\ln n)^2}{n^3} + \frac{(2g(a_0)-1)\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right), \tag{4}$$

$$\begin{aligned}\frac{1}{a_n} &= n + \ln n + g(x) + \frac{\ln n}{n} + \frac{(-\frac{1}{2}+g)}{n} - \frac{1}{2}\frac{(\ln n)^2}{n^2} + \\ &\frac{(\frac{3}{2}-g)\ln n}{n^2} + \left(\frac{3}{2}g - \frac{1}{2}g^2 - \frac{5}{6}\right)\frac{1}{n^2} + \frac{1}{3}\frac{(\ln n)^3}{n^3} + \\ &(-2+g)\frac{(\ln n)^2}{n^3} + \left(\frac{19}{6} - 4g + g^2\right)\frac{\ln n}{n^3} + o\left(\frac{\ln n}{n^3}\right).\end{aligned}$$

Proof. The sequence $x_n = \frac{1}{a_n}$, $n \geq 1$, satisfies the recurrence relation $x_{n+1} = h(x_n)$, where $h(x) = x + 1 + \frac{1}{x-1}$, for $x \in (1, \infty)$. If we define $r(x_1) = \lim_{n\to\infty} y_n$ with $y_n = x_n - n - ln\ n$, clearly $g(x) = r(1/x)$ for all $x \in I$. Since all the properties of r transfer to g in a corresponding way, we prefer to work with the function r instead of g. Directly from the recurrence relation for x_n we easily see that x_n is a strictly increasing sequence, $x_2 \geq 4, (h(1,\infty)) = [4,\infty))$, and we get

$$x_{n+1} = x_2 + n - 1 + \sum_{k=2}^{n} \frac{1}{x_k - 1},\ n \geq 2. \tag{5}$$

From this we obtain that $x_n \geq n+2$ for all $n \geq 2$. This shows, in particular, that the limit defining r exists, since y_n is a decreasing sequence:

$$y_n - y_{n+1} = \ln(1+\frac{1}{n}) - \frac{1}{x_n - 1} > \frac{1}{n+1} - \frac{1}{x_n - 1} \geq 0,\ n \geq 2.$$

Secondly, going back to (5), the next better estimation from above of x_n results:

$$x_{n+1} \leq x_2 + n - 1 + \sum_{k=2}^{n} \frac{1}{k+1} < x_2 + n - 1 + \ln(n+1) - \ln 2,\ n \geq 2. \tag{6}$$

Since for $u > v \geq 2$ or $1 < u < v \leq 2$, we get $h(u) > h(v) \geq 4$, and then $h(h(u)) > h(h(v) \geq 4$, a simple induction argument shows that r is decreasing on $(1, 2]$ and increasing on $[2, \infty)$. Therefore, in order to prove that r has finite values, it is enough to show that $r(2) > 0$. Hence, if $x_1 = 2$, (6) becomes

$$x_n \leq n + \omega + \ln n,\ n \geq 2, \tag{7}$$

where $\omega = 2 - \ln 2 > 1$. Using (7) in (5), we obtain

$$x_{n+1} \geq n + 3 + \sum_{k=2}^{n} \frac{1}{k - 1 + \omega + \ln k},\ n \geq 2.$$

This implies that for $n \geq 2$

$$y_{n+1} \geq 2 - \ln(n+1) + \sum_{k=1}^{n-1} \frac{1}{k + \omega + \ln(k+1)} > 2 - \ln(n+1) + \int_1^n \frac{dx}{x + \omega + \ln(x+1)}.$$

Since $\dfrac{1}{x+\omega+\ln(x+1)} > \dfrac{1}{(x+\omega)} - \dfrac{\ln(x+1)}{(x+\omega)^2}$ on the interval $[1,\infty)$, we can continue the above sequence of inequalities as follows:

$$\begin{aligned} y_{n+1} &\geq 2-\ln(n+1)+\int_1^n \frac{dx}{(x+\omega)} - \int_1^n \frac{\ln(x+1)dx}{(x+\omega)^2} = 2-\ln(1+\omega)+ \\ &+\ln(\frac{n+\omega}{n+1}) - \int_1^n \frac{\ln(x+1)dx}{(x+\omega)^2} > 2-\ln(1+\omega) - \int_1^\infty \frac{\ln(x+1)dx}{(x+\omega)^2} = \\ &= 2+\frac{2\ln 2}{\omega^2-1} - \frac{\omega}{\omega-1}\ln(1+\omega). \end{aligned}$$

Since $\ln(1+\omega) = \ln\ 2(1+\frac{\omega-1}{2}) < \ln 2 + \frac{\omega-1}{2} = \frac{3-\omega}{2}$, we obtain from the above computation that

$$r(2) = \lim_n y_{n+1}(2) \geq \frac{(\omega-1)(\omega+4)}{2(\omega+1)} > 0.$$

Hence we have proved the second part of the statement (ii) in Theorem 3.

We next look at the sequence of the derivatives of the functions $x_n(x) = h^n(x)(x_1 = x)$, where $h^{n+1}(x) = h(h^n(x))$, $n \geq 1$. Since $h'(x) = 1 - \frac{1}{(x-1)^2}$, and $(h^n)'(x) = h'(h^{n-1}(x))h'(h^{n-2}(x))...h'(x)$, we get

$$y'_n = x'_n = \prod_{k=1}^{n-1}\left(1-\frac{1}{(x_k-1)^2}\right),\ n \geq 2. \tag{8}$$

Using the inequality $x_n \geq n+2$, $n \geq 2$, the product appearing in (8) is absolutely convergent. Therefore the sequence $y_n(x) = y_n(2) + \int_2^x y'_n(t)dt$ converges to $r(x) = r(2) + \int_2^x \prod_{k=1}^{\infty}(1 - \dfrac{1}{(x_k(t)-1)^2})dt$. In particular, this shows that r is continuously differentiable. In order to complete the proof of (i), let us observe that

$$\begin{aligned} r(h(x)) &= \lim_n\ y_n(h(x)) = \lim_n\ x_{n+1}(x) - n - \ln n = \\ &= \lim_n\ x_n(x) + 1 + \frac{1}{x_n-1} - n - \ln n = r(x)+1. \end{aligned}$$

Hence $g(f(x)) = r(1/f(x)) = r(h(1/x)) = r(1/x)+1$ and $g(1-x) = g(f(1-x))-1 = g(f(x))-1 = g(x)$, for $x \in I$, which completes the proof of (i). Because

$$r'(x) = \prod_{k=1}^{\infty}\left(1-\frac{1}{(x_k(x)-1)^2}\right) = \frac{x(x-2)}{(x-1)^2}\prod_{k=2}^{\infty}\left(1-\frac{1}{(x_k(x)-1)^2}\right), \tag{9}$$

it is easy to see that $r'(x) > 0$ for $x > 2$ and $r'(x) < 0$ for $1 < x < 2$. This completes the proof of (ii).

To get (iii) we can use (i) to obtain $g'(f(x))f'(x) = g'(x)$, and hence by the change of variable formula,

$$\begin{aligned} \xi(f(A)) &= \int_{f(A)} d\xi(x) = \int_{f(A)} g'(x)dx = \int_{f(A)} g'(f(x))f'(x)dx \\ &= \int_{f(A)} g'(f(x))f'(x)dx = \int_A g'(x)dx = \int_A d\xi(x) = \xi(A). \end{aligned}$$

In order to prove (iv), let us compute $\ln(r'(x))$ for $x > 2$, using formula (9) and the recursive relation:

$$\begin{aligned}
\ln(r'(x)) &= \ln\left(\prod_{k=1}^{\infty}\left(1-\frac{1}{(x_k(x)-1)^2}\right)\right)\\
&= \ln\left(\lim_n \prod_{k=1}^{n}\left(1-\frac{1}{x_k(x)-1}\right)\prod_{k=1}^{n}\left(1+\frac{1}{x_k(x)-1}\right)\right)\\
&= \lim_n\left(\sum_{k=1}^{n}\ln\left(1-\frac{1}{x_k(x)-1}\right)+\ln\left(\prod_{k=1}^{n}\frac{x_k(x)}{x_k(x)-1}\right)\right)\\
&= \lim_n\left(-\sum_{k=1}^{n}\sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j+\ln\left(\prod_{k=1}^{n}\frac{x_{k+1}(x)}{x_k(x)}\right)\right).
\end{aligned}$$

After we interchange the sums, using (5) we can continue the above computation as follows:

$$\begin{aligned}
\ln(r'(x)) &= \lim_n\left(\ln(x_{n+1}(x))-\ln\, x-\sum_{k=1}^{n}\frac{1}{x_k(x)-1}-\sum_{j=2}^{\infty}\sum_{k=1}^{n}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j\right)\\
&= -\ln\, x+\lim_n\left(\ln(x_{n+1}(x))-x_{n+1}(x)+n+x-\sum_{j=2}^{\infty}\sum_{k=1}^{n}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j\right)\\
&= x-1-\ln x-\lim_n\left(x_{n+1}(x)-(n+1)-\ln(n+1)+\ln\left(\frac{n+1}{x_{n+1}(x)}\right)\right)\\
&\quad -\lim_n\left(\sum_{j=2}^{\infty}\sum_{k=1}^{n}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j\right).
\end{aligned}$$

Because the double sum $\displaystyle\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j$ is absolutely convergent we can interchange the limit sign with the sum sign in the above computation, and using the definition of r we obtain the following differential equation in r:

$$\ln(r'(x)) = x-1-\ln x-r(x)-\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j,$$

or

$$r'(x)\exp(r(x)) = \frac{1}{x}\exp(x-1-R(x)), \tag{10}$$

where $R(x) = \displaystyle\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j}\left(\frac{1}{x_k(x)-1}\right)^j$. Integrating (10), we obtain a formula which gives us another way of approximating the values of r:

$$r(x) = \ln\left(C+\int_2^x\frac{1}{t}\exp\left(t-1-R(t)\right)dt\right),\ \ x>2. \tag{11}$$

In terms of the function g and the sequence $\{a_n\}$, after a change of variable, the formula (11) becomes

$$g(x) = \ln\left(C+\int_x^{\frac{1}{2}}\frac{1}{u}\exp\left(\frac{1}{u}-1-G(u)\right)du\right),\ \ x\in(0,1/2),$$

where $G(u) = R(1/u) = \displaystyle\sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j}\left(\frac{1}{x_k(1/u)-1}\right)^j = \sum_{j=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j}\left(\frac{a_k(u)}{1-a_k(u)}\right)^j$, and (iv) is proved.

To prove (v), we apply several times part (ii) of Cesàro's Lemma. First we take $u_n = x_n(x) - n - \ln\ n - r(x)$ and $v_n = (1/n)\ln n$:

$$\lim_n \frac{n(x_n(x) - n - \ln n - r(x))}{\ln n} = \lim_n \frac{\frac{1}{x_n - 1} - \ln(1 + \frac{1}{n})}{\frac{\ln(n+1)}{n+1} - \frac{\ln n}{n}} = 1.$$

Using the same technique we can compute the other terms in (11), and (10) is easily obtained from (11). □

We point out that there are cases when it is easy to determine expansions as in (4) for all $k \geq 2$. For example, if $f(x) = x/(1+x)$, then $\{a_n\}_n$ has the expansion of the form

$$a_n = \sum_{j=0}^{m} \frac{(-1)^j}{n^{j+1} a_0^j} + o\left(\frac{1}{n^{m+1}}\right), \quad n \geq 1, \quad a_0 \in (0, \infty).$$

That can be seen easily by linearizing the recurrence $a_{n+1} = f(a_n)$ in the following way: replace $\dfrac{1}{a_n}$ by b_n. We obtain the linear equation $b_{n+1} = b_n + 1$, which obviously produces $a_n = \dfrac{1}{n + a_0^{-1}}$, from which we infer the previous approximation.

On the other hand, if $f(x) = \sin x$, we computed using Theorem 2 the following expansion:

$$a_n = \frac{\sqrt{3}}{\sqrt{n}} - \frac{3\sqrt{3}}{10} \frac{\ln n}{n\sqrt{n}} + \frac{9\sqrt{3}}{50} \frac{\ln n}{n^2\sqrt{n}} + o\left(\frac{\ln n}{n^2\sqrt{n}}\right),$$

where the coefficients do not seem to depend on the initial value of the sequence.

3. Almost Doubly-Exponential Sequences

Aho and Sloane [**1**] considered the sequences of the form $a_{n+1} = a_n^2 + g_n$, where $a_n > 0$, $|g_n| \leq a_n/4$, $a_n \geq 1$ and $|\log(a_{n+1}a_n^{-2})|$ is decreasing, for $n \geq n_0$. They proved that under these conditions, there exists a constant k such that $a_n =$ *nearest integer to* k^{2^n}. Obviously, the sequence of Theorem 3 is not among the ones considered by Aho and Sloane, since it does not satisfy the mentioned conditions. In the spirit of [**1**], relaxing the conditions, using a somewhat different method, we prove the next theorem, involving what we call *almost doubly-exponential* recurrences.

Theorem 4. *Let the sequence of positive integers $a_{n+1} = a_n^2 + g_n$, satisfying $-a_n + 1 < g_n < a_n$, $a_n > 1$ and $\left|\log\left(a_{n+1}a_n^{-2}\right)\right|$ is decreasing (for $n \geq n_0$). Then there exists α such that $a_n = \lfloor \exp(2^n\alpha) \rfloor$, or $a_n = \lfloor \exp(2^n\alpha) \rfloor + 1$ (for $n \geq n_0$).*

Proof. Since the entire proof refers to $n \geq n_0$, we may as well assume that $n_0 = 0$. The proof uses some ideas of [**1**] and [**6**]. Let $u_n := \log a_n$, and $\delta_n := \log(g_n a_n^{-2} + 1)$. Thus $u_{n+1} = 2u_n + \delta_n$. Iterating we get

$$u_n = 2^n u_0 + 2^n \sum_{k=0}^{n-1} \delta_k 2^{-k-1}.$$

The series $\alpha := u_0 + \sum_{k=0}^{\infty} \delta_k 2^{-k-1}$ is absolutely convergent since $|\delta_k| < \log(1 + a_k^{-1}) < \log 2$. Taking $r_n := 2^n\alpha - u_n$, we get that $a_n = \exp(u_n) = \exp(2^n\alpha - r_n)$. Now,

$$\exp(2^n\alpha) = a_n \exp(r_n), \quad \text{and}$$

(12)
$$r_n = 2^n \sum_{k=n}^{\infty} \delta_k 2^{-k-1} = \sum_{k=0}^{\infty} \delta_{k+n} 2^{-k-1}.$$

Since $\left|\log(a_{n+1}a_n^{-2})\right| = \left|\log(g_n a_n^{-2} + 1)\right| = |\delta_n|$ is decreasing, we get

$$|r_n| \leq \sum_{k=0}^{\infty} |\delta_{k+n}| 2^{-k-1} \leq |\delta_n| \sum_{k=0}^{\infty} 2^{-k-1} = |\delta_n|$$

which implies

$$a_n \exp(-|\delta_n|) \le \exp(2^n\alpha) \le a_n \exp(|\delta_n|). \tag{13}$$

We use now the definition of δ_n, and deduce

$$\begin{aligned} \exp(\delta_n) &= g_n a_n^{-2} + 1, \\ \exp(-\delta_n) &= (g_n a_n^{-2} + 1)^{-1}. \end{aligned} \tag{14}$$

Therefore, using (13) and (14), if $\delta_n > 0$, then

$$a_n - \exp(2^n\alpha) \le a_n - a_n \exp(-\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)^{-1}\right), \tag{15}$$

$$a_n - \exp(2^n\alpha) \ge a_n - a_n \exp(\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)\right) = -g_n a_n^{-1}. \tag{16}$$

Now, in (15) to have $a_n\left(1 - (g_n a_n^{-2} + 1)^{-1}\right) < 1$, it is necessary to have $(g_n a_n^{-2}+1)^{-1} > 1-1/a_n$ which in turn is equivalent to $g_n < \dfrac{a_n^2}{a_n - 1} = a_n + 1 + \dfrac{1}{a_n - 1}$. The last inequality is true since $g_n < a_n$. In (16) to have $-g_n a_n^{-1} > -1$, it is necessary to have $g_n < a_n$.

If $\delta_n < 0$, by (13) and (14), then

$$a_n - \exp(2^n\alpha) \le a_n - a_n \exp(\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)\right) = -g_n a_n^{-1}, \tag{17}$$

$$a_n - \exp(2^n\alpha) \ge a_n - a_n \exp(-\delta_n) = a_n \left(1 - (g_n a_n^{-2} + 1)^{-1}\right). \tag{18}$$

Now, in (17), $-g_n a_n^{-1} < 1$ is equivalent to $g_n > -a_n$, and the last inequality is certainly true, since $g_n > -a_n + 1$. In (18) to have $a_n\left(1 - (g_n a_n^{-2} + 1)^{-1}\right) > -1$, it is necessary to have $g_n a_n^{-2} + 1 > \dfrac{a_n}{a_n + 1} = 1 - \dfrac{1}{a_n + 1}$. That is equivalent to $g_n > \dfrac{-a_n^2}{a_n + 1} = -a_n + 1 - \dfrac{1}{a_n + 1}$, which is certainly true, as g_n is an integer, $a_n > 1$ and $g_n > -a_n + 1$.

Thus, we obtain, in any case, that $|a_n - \exp(2^n\alpha)| < 1$, which implies (since a_n is an integer) that $a_n = \lfloor \exp(2^n\alpha) \rfloor$, or $a_n = \lfloor \exp(2^n\alpha) \rfloor + 1$. □

Remark 5. *The previous theorem does not consider the case of $g_n = -a_n + 1$ (the lower bound). However, in that case we get $a_{n+1} = a_n^2 - a_n + 1$, which was dealt with by Aho and Sloane (Recurrence* 2.4*), if $a_1 = 2$, being transformed into a recurrence satisfying their conditions, deriving the solution $\lfloor k^{2^n} + \frac{1}{2} \rfloor$, for some real number k.*

Consider now that case of $a_n < g_n < 2a_n$ in the recurrence $a_{n+1} = a_n^2 + g_n$, $a_n > 1$ positive integers. Let $g_n' = g_n - a_n$. Thus, $0 < g_n' < a_n$ and the recurrence can be written as

$$a_{n+1} = a_n^2 + a_n + g_n'.$$

Let $b_n = a_n + \frac{1}{2}$ and $h_n = g_n' - \frac{3}{4} = g_n - a_n - \frac{3}{4}$. It follows that

$$b_{n+1} = b_n^2 + h_n, \text{ with } -\frac{3}{4} < h_n < a_n - \frac{3}{4} < a_n,$$

which is of the first type, but (beware!) this sequence does not consist of integers. We start with one observation: since $a_n < g_n$, it follows that $g_n - a_n \ge 1$, therefore $h_n \ge \frac{1}{4}$, so h_n satisfies $0 < h_n < a_n$.

Let $u_n := \log b_n$, and $\delta_n := \log(h_n b_n^{-2} + 1)$. If $|\log(b_{n+1} b_n^{-2}|$ is decreasing, the same technique as before renders, since $h_n > 0$,

$$\begin{aligned} b_n - \exp(2^n\beta) &\le b_n(1 - (h_n b_n^{-2} + 1)^{-1}), \\ b_n - \exp(2^n\beta) &\ge -h_n b_n^{-1}, \end{aligned}$$

where $\beta := u_0 + \sum_{k=0}^{\infty} \delta_k 2^{-k-1}$. Moreover, $b_n(1 - (h_n b_n^{-2} + 1)^{-1}) < 1$ if and only if $\dfrac{b_n - 1}{b_n} < \dfrac{1}{h_n b_n^{-2} + 1}$. This is equivalent to $h_n < \dfrac{b_n^2}{b_n - 1} = b_n + 1 + \dfrac{1}{b_n - 1}$, which is certainly true as $h_n < a_n < a_n + \frac{1}{2} = b_n$.

Furthermore, since $-h_n b_n^{-1} > -1$, then

$$-\frac{3}{2} < a_n - \exp\left(2^n\beta\right) < \frac{1}{2}.$$

The right hand side inequality is improved by the simple observation that since $\delta_k > 0$, then $2^n\beta > u_n$, therefore, $\exp(2^n\beta) > b_n = a_n + \frac{1}{2}$, which implies

$$-\frac{3}{2} < a_n - \exp\left(2^n\beta\right) < -\frac{1}{2}$$

and so,

$$a_n < \exp(2^n\beta) - \frac{1}{2} < a_n + 1.$$

To cover the whole range $-a_n + 1 < g_n < 2a_n$, it suffices to study the case of $g_n = a_n$. In that case, we get the recurrence of positive integers $a_{n+1} = a_n^2 + a_n$. Taking $b_n = a_n + 1/2$, we get

$$b_{n+1} = b_n^2 - \frac{3}{4},$$

which was dealt with by Aho and Sloane, if $b_1 = \frac{3}{2}$, obtaining $b_n = \frac{3}{2} + \left\lfloor k^{2^n} + \frac{3}{2}\right\rfloor$, $n \geq 3$, for some real k.

Thus, we have proved

Theorem 6. *Let the recurrence of positive integers $a_{n+1} = a_n^2 + g_n$, where $a_n < g_n < 2a_n$, $a_n > 1$ (if $n \geq n_0$). Also assume that $|\log\left((a_{n+1} + 1/2)(a_n + 1/2)^{-2}\right)|$ is decreasing. Then there exists a real number β such that*

$$a_n = \left\lfloor \exp(2^n\beta) - \frac{1}{2}\right\rfloor, \text{ if } n \geq n_0.$$

If $a_{n+1} = a_n^2 + a_n$ and $a_1 = 1$, then

$$a_n = \left\lfloor \exp(2^n\beta) + \frac{5}{2}\right\rfloor, \text{ if } n \geq 3.$$

Certainly the theorem can be further extended by taking various other intervals for g_n and imposing the restrictive decreasing property on a_n.

The sequence g_n may or may not depend on a_n. If $g_n = a_n - 2a_n^2$, we end up with a recurrence of the form $a_{n+1} = f(a_n)$, where $f(x) = x - x^2$. Obviously, in this case Theorem 4 is not true, since the inequality imposed on g_n does not hold. But this case was dealt with by Theorem 3.

Can we relax the conditions of Theorem 4 and Theorem 6 even further? The answer is yes, but the result is not that accurate. Let the recurrence of positive integers $a_{n+1} = a_n^2 + h_n$ with $|h_n| < (1+\epsilon)a_n$, $a_n \geq 1$, where $\epsilon > 0$ is a fixed parameter. In the same manner as before, we denote by $\delta_n(\epsilon) = \log(h_n a_n^{-2} + 1)$ and $u_n = \log a_n$. The series $\alpha(\epsilon) = u_0 + \sum_{k=0}^{\infty} \delta_k(\epsilon)2^{-k-1}$ is convergent since $-\log(2+\epsilon) \leq \log(1 - \frac{1+\epsilon}{a_k}) < \delta_k(\epsilon) < \log(1 + \frac{1+\epsilon}{a_k}) < \log(2+\epsilon)$, for k sufficiently large so that $a_k > 1+\epsilon$. Taking $r_n = 2^n\alpha - u_n$, we get that $a_n = \exp(u_n) = \exp(2^n\alpha - r_n)$. We did not impose the decreasing property on $|\delta_n(\epsilon)|$, so we can only infer at this stage that

$$-\log(2+\epsilon) \leq r_n = \sum_{k=0}^{\infty} \delta_{k+n}(\epsilon)2^{-k-1} \leq \log(2+\epsilon), \tag{19}$$

using the double inequality on $\delta_n(\epsilon)$.

With a bit more work, we conclude

Proposition 7. *Let $a_{n+1} = a_n^2 + h_n$ with $|h_n| < (1+\epsilon)a_n$, $a_n \geq 1$, where $\epsilon \geq 0$ is a fixed parameter. Then there exists a constant α such that*

$$\frac{1}{2+\epsilon}\exp(2^n\alpha) \leq a_n \leq (2+\epsilon)\exp(2^n\alpha),$$

if n is sufficiently large so that $a_n > 1+\epsilon$.

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