Remarks on a Sequence of Minimal Niven Numbers*

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Abstract. In this short note we introduce two new sequences defined using the sum of digits in the representation of an integer in a certain base. A connection to Niven numbers is proposed and some results are proven.

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1 Introduction

A positive integer n is called a *Niven number* (or a *Harshad number*) if it is divisible by the sum of its decimal digits. For instance, 2007 is a Niven number since 9 divides 2007. A q-*Niven number* is one which is divisible by the sum of its base q digits (incidentally, 2007 is also a 2-Niven number). Niven numbers have been extensively studied by various authors (see [1,2,3,4,5,7,8,10], just to cite a few of the most recent works). We let $s_q(k)$ be the sum of digits of k in base q.

In this note, we define two sequences in relation to q-Niven numbers. For a fixed but arbitrary $k \in \mathbb{N}$ and a base $q \geq 2$, we ask if there exists a q-Niven number whose sum of its digits is precisely k. Therefore it makes sense to define a_k to be the smallest positive multiple of k such that $s_q(a_k) = k$. In other words, a_k is the smallest Niven number whose sum of the digits is a given positive integer k (trivially, for every k such that $1 \leq k < q$ we have $a_k = k$). We invite the reader to check that, for instance, $a_{12} = 48$ in base 10.

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In [6] we remarked that q-Niven numbers with only 0's or 1's in their q-base representation, with a fixed sum of digits, do exist. So, we define b_k as the smallest positive multiple of k, which written in base q has only 0's or 1's as digits, and in addition $s_q(b_k) = k$. Obviously, a_k and b_k depend on q, but we will not make this explicit to avoid complicating the notation. Clearly, in base 2 we have $a_k = b_k$ for all k but for $q > 2$ we actually expect a_k to be a lot smaller than b_k.

2 The Results

We start with a simple argument (which is also included in [6]) that shows that the above sequences are well defined. First we assume that k satisfies $\gcd(k, q) = 1$. By Euler's theorem, we can find t such that $q^t \equiv 1 \pmod{k}$, and then define

$$K = 1 + q^t + q^{2t} + \cdots + q^{(k-1)t}.$$

Obviously, $K \equiv 0 \pmod{k}$, and so $K = kn$ for some n and also $s_q(K) = k$. Hence, in this case, K is a Niven number whose digits in base q are only 0's and 1's and whose sum is k. This implies the existence of a_k and b_k.

If k is not coprime to q, we can assume that $k = ab$ where $\gcd(b, q) = 1$ and a divides q^n for some $n \in \mathbb{N}$. As before, we can find a multiple of b, say K, such that $s_q(K) = b$. Let $u = \max\{n, \lceil \log_q K \rceil\} + 1$, and define

$$K' = (q^u + q^{2u} + \cdots + q^{ua})K.$$

Certainly $k = ab$ is a divisor of K' and $s_q(K') = ab = k$. Therefore, a_k and b_k are well defined for every $k \in \mathbb{N}$.

However, this argument gives a large upper bound, namely of size $\exp(O(k^2))$ for a_k. In the companion paper [6], we present constructive methods by two different techniques for the binary and nonbinary cases, respectively, yielding sharp upper bounds for the numbers a_k and b_k. Here we point out a connection with the q-Niven numbers. The binary and decimal cases are the most natural cases to consider. The table below describes the sequence of minimal Niven numbers a_k for bases $q = 2, 3, 5, 7, 10$, where $k = 2, \ldots, 25$.

We remark that if m is the minimal q-Niven number corresponding to k, then $q - 1$ must divide $m - s_q(m) = kc_k - k = (c_k - 1)k$. This observation turns out to be useful in the calculation of a_k for small values of k. For instance, in base ten, a_{17} can be established easily by using this simple property: 9 has to divide $c_{17} - 1$ and so we check for c_{17} the values 10, 19, and see that 28 is the first integer of the form $9m + 1$ ($m \in \mathbb{N}$) that works.

In some cases, one can find a_n explicitly, as our next result shows. In [6] we proved the following result.

Lemma 1. *If* $q > 2$, *then*

$$a_{q^m} = q^m \left(2q^{\frac{q^m - 1}{q-1}} - 1 \right).$$

If $q = 2$, *then* $a_{2^m} = 2^m (2^{2^m} - 1)$.

Table 1. Values of a_k in various bases

k	*base* 2	*base* 3	*base* 5	*base* 7	*base* 10
2	6	4	6	8	110
3	21	15	27	9	12
4	60	8	8	16	112
5	55	25	45	65	140
6	126	78	18	12	24
7	623	77	63	91	133
8	2040	80	24	32	152
9	1503	1449	117	27	18
10	3070	620	370	40	190
11	3839	1133	99	143	209
12	16380	2184	324	48	48
13	16367	3887	949	325	247
14	94206	4130	574	1022	266
15	96255	30615	4995	195	195
16	1048560	6560	624	832	448
17	483327	19601	2873	629	476
18	524286	177138	3114	342	198
19	1040383	58805	6099	1273	874
20	4194300	137780	15620	1700	3980
21	5767167	354291	12369	9597	399
22	165 15070	347732	12474	2398	2398
23	16252927	529253	31119	6509	1679
24	134217720	1594320	15624	2400	888
25	66584575	1417175	781225	10975	4975

The first part of the following lemma is certainly known, but we include a short proof for completeness.

Lemma 2. *Let* $q \geq 2$ *and* k, n *be positive integers. Then* $s_q(nk) \leq s_q(k)s_q(n)$. *In particular,* $k = s_q(a_k) \leq s_q(k)\, s_q(a_k/k)$. *A similar inequality holds for* b_k, *and both such inequalities are sharp regardless of the base* q.

Proof. Write

$$n = \sum_{i=0} n_i q^i, \qquad \text{and} \qquad k = \sum_{j=0} k_j q^j, \qquad \text{where } n_i, k_j \in \{0, 1, \ldots, q-1\},$$

for all indices i and j. Certainly, the product $nk = \sum_{i=0} \sum_{j=0} n_i k_j q^{i+j}$ is not necessarily the base q expansion of nk, as a certain value of $i + j$ may occur multiple times, or some products $n_i k_j$ may exceed q. However, $s_q(nk) \leq \sum_{i=0} \sum_{j=0} n_i k_j = s_q(k)s_q(n)$, which implies the first assertion.

Let us show that the inequalities are sharp in every base q. If $q = 2$, then letting $k = 2^m$, we get, by Lemma 1, that $a_{2^m} = 2^m(2^{2^m} - 1)$, $s_2(a_k) = 2^m$, $s_2(k) = 1$, and $s_2(a_k/k) = 2^m$, which shows that indeed $s_2(a_k) = s_2(k)s_2(a_k/k)$.

Similarly, Lemma 1 implies that this inequality is sharp for an arbitrary base q as well. □

Let us look at the base 2 case. In [6], we have shown that

Theorem 3. *For all integers $k = 2^i - 1 \geq 3$, we have*

$$a_k \leq 2^{k+k^-} + 2^k - 2^{k-i} - 1, \tag{1}$$

where k^- is the least positive residue of $-k$ modulo i. Furthermore, the bound (1) *is tight when $k = 2^i - 1$ is a Mersenne prime.*

We extend the previous result in our next theorem, whose proof is similar to the proof of Theorem 3 in [6] using obvious modifications for the second claim, however we are going to include it here for the convenience of the reader. It is worth mentioning that, as a corollary of this theorem, the value of a_k is known for every k which is an even perfect number (via the characterization of the even perfect numbers due to the ancient Greeks, see Theorem 7.10 in [9]).

Theorem 4. *For all integers $k = 2^s(2^i - 1) \geq 3$, with $i, s \in \mathbb{Z}$, $i \geq 2$, $s \geq 0$, we have*

$$a_k \leq 2^s(2^{k+k^-} + 2^k - 2^{k-i} - 1), \tag{2}$$

where k^- is the least nonnegative residue of $-k$ modulo i. Furthermore, the bound (2) *is tight when $2^i - 1$ is a Mersenne prime.*

Proof. For the first claim, it suffices to show that the sum of binary digits of the upper bound on (2) is exactly k, and also that this number is a multiple of k.

Indeed, from the definition of k^-, we find that $k + k^- = ia$ for some positive integer a. Since

$$\begin{aligned} 2^{k+k^-} + 2^k - 2^{k-i} - 1 &= 2^{k-i}(2^i - 1) + 2^{ia} - 1 \\ &= (2^i - 1)(2^{k-i} + 2^{i(a-1)} + 2^{i(a-2)} + \cdots + 1), \end{aligned}$$

we get that $2^s(2^{k+k^-} + 2^k - 2^{k-i} - 1)$ is divisible by k.

For the sum of the binary digits we have

$$\begin{aligned} s\left(2^{k+k^-} + 2^k - 2^{k-i} - 1\right) &= s\left(2^{k+k^- -1} + \cdots + 2 + 1 + 2^k - 2^{k-i}\right) \\ &= s\left(2^{k+k^- -1} + \cdots + 2^k + \cdots + \widehat{2^{k-i}} + \cdots + 2 + 1 + 2^k\right) \\ &= s\left(2^{k+k^-} + 2^{k-1} + \cdots + \widehat{2^{k-i}} + \cdots + 2 + 1\right) \\ &= k, \end{aligned}$$

where $\hat{t}$ means that t is missing in that sum. The first claim is proved.

We now consider that $p = 2^i - 1$ is a Mersenne prime. Then we need to show that the right hand side of (2) is the smallest number that satisfies the conditions mentioned above. The divisibility condition implies that $a_k = 2^s x$ for some $x \in \mathbb{N}$. We need to show that $x = 2^{k+k^-} + 2^k - 2^{k-i} - 1$, or in other words,

x is the smallest number that has the sum of its digits in base 2 equal to k and it is divisible by p.

We know that $a_k \geq 2^k - 1$. Let us denote by m the first positive integer with the property that

$$2^{k+m} - 1 \equiv 2^{j_1} + 2^{j_2} + ... + 2^{j_m} \pmod{p} \tag{3}$$

for some $0 \leq j_1 < ... < j_m \leq k+m-2$. Notice that any other $m' > m$ will have this property and if we denote by $y = 2^{j_1}+2^{j_2}+...+2^{j_m}$ the $a_k = 2^s(2^{k+m}-1-y)$ where j_1, j_2,..., j_m are chosen to maximize y. Because

$$x = 2^{k+k^-+1} - 1 - (2^{k-i} + 2^{k+k^- -1} + 2^{k+k^- -2} + ... + 2^k) \equiv 0 \pmod{p}$$

we deduce that $m \leq k^-+1$. Let us show that $m < k^-+1$ leads to a contradiction. It is enough to show that $m = k^-$ leads to a contradiction. $2^{k+k^-} \equiv 2^{ia} \equiv 1 \pmod{p}$. Hence $0 \equiv 2^{j_1} + 2^{j_2} + ... + 2^{j_m} \pmod{p}$. Because $2^i \equiv 1 \pmod{p}$, we can reduce all powers 2^j of 2 modulo p to powers with exponents less than or equal to $i-1$. We get at most $m \leq i-1$ such terms. But in this case, the sum of at least one and at most $i-1$ distinct members of the set $\{1, 2, \ldots, 2^{i-1}\}$ is positive and less than the sum of all of them, which is p. So, the equality (3) is impossible in this case.

Therefore $m = k^- + 1$ and one has to choose j_1, j_2,...,j_m in order to maximize y. This means $j_m = k+m-1$, $j_{m-1} = k+m-2, \ldots$, and finally j_1 has to be chosen in such a way it is the greatest exponent less than k such that $2^{k+m} - 1 - y \equiv 0 \pmod{p}$. Since $j_1 = k - i$ satisfies this condition and because the multiplicative index of 2 $\pmod{p}$ is i this choice is precisely the value for j_1 which maximizes y. □

Next, we find by elementary methods an upper bound on a_k.

Theorem 5. *If k is a 2-Niven number, then*

$$a_k \leq k\frac{2^{i_s(k/s+1)} - 1}{2^{i_s} - 1},$$

where $s = s_2(k)$ and i_s is the largest nonzero binary digit of k. Moreover, the equality $s_2(a_k) = s_2(k)s_2(a_k/k)$ holds for at least

$$2\log 2\frac{N}{\log N} + O\left(\frac{N}{(\log N)^{9/8}}\right)$$

integers $k \leq N$.

Proof. The observation allowing us to construct a multiple kd_k of k such that $s_2(kd_k) = k$ out of any 2-Niven number k, is to observe that we may choose d_k such that if $s_2(kd_k) = k$, then $s_2(d_k) = k/s_2(k)$. Thus, if

$$k = \sum_{i=0}^{N} k_i 2^i \qquad \text{and} \qquad d_k = \sum_{j=0}^{K} n_j 2^j,$$

then $k\,d_k = \sum_{i=0}^{N}\sum_{j=0}^{K} k_i n_j 2^{i+j}$. The equality holds if this is indeed the binary expansion of $k\,d_k$, that is, if $i+j$ are all distinct for all choices of i and j such that $k_i n_j \neq 0$.

This argument gives us a way to generate d_k. Let $k_{i_1}, k_{i_2}, \ldots, k_{i_s}$ be all the nonzero binary digits of k, where $s = s_2(k)$. Put $m = k/s$. Recall that d_k must be odd, and so the least nonzero digit of d_k is 1. We shall define a sequence of disjoint sets in the following way. Set $d_1 = 0$, and

$$A_1 = \{i_1, i_2, \ldots, i_s\}.$$

Now, let $d_2 = \min\{d \in \mathbb{N} \mid d - i_1 + i_\ell \notin A_1, \ell = 1, \ldots, s\} - i_1$ and set

$$A_2 = \{d_2 + i_1, d_2 + i_2, \ldots, d_2 + i_s\}.$$

Next, let $d_3 = \min\{d \in \mathbb{N} \mid d - i_1 + i_\ell \notin A_1 \cup A_2, \ell = 1, \ldots, s\} - i_1$ and set

$$A_3 = \{d_3 + i_1, d_2 + i_2, \ldots, d_3 + i_s\}.$$

Continue the process until we reach $d_m = \min\{d \in \mathbb{N} \mid d - i_1 + i_\ell \notin A_1 \cup A_2 \cup \cdots \cup A_{m-1}, \ell = 1, \ldots, s\} - i_1$ and set

$$A_m = \{d_m + i_1, d_m + i_2, \ldots, d_m + i_s\}.$$

Further, we define

$$d_k = 2^{d_1} + 2^{d_2} + 2^{d_3} + \cdots + 2^{d_m}. \tag{4}$$

Next, observe that

$$k\,d_k = \sum_{\ell=1}^{s} 2^{i_l} \sum_{p=1}^{m} 2^{d_p} = \sum_{r=1}^{m} \sum_{t \in A_r} 2^t,$$

and so the binary sum of digits of $k\,d_k$ is $s_2(k\,d_k) \leq \sum_{r=1}^{m} \sum_{t \in A_r} 1 = ms = k$, since the cardinality of each partition set A_r is s.

Regarding the bound on a_k, the worst case that can arise would be to take $d_j = j i_s$ at every step in the construction of the sequence of sets A_j. Thus, an upper bound for a_k is given by

$$a_k \leq 1 + 2^{i_s} + \cdots + 2^{m \cdot i_s} = \frac{2^{i_s(m+1)} - 1}{2^{i_s} - 1}.$$

We now observe that if the equality $s_2(a_k) = s_2(k)s_2(a_k/k)$ holds, since $k = s_2(a_k)$, then k is a 2-Niven number. Finally, the last estimate follows from the previous observation together with Theorem D of [7] concerning the counting function of the 2-Niven numbers. □

Let us consider an example to illustrate the approach of Theorem 5. Let $n = 34 = 2^1 + 2^5$. Thus, $s = 2$, $i_1 = 1, i_2 = 5$. Now, the sequence of the sets A_i, where $i = 1, \ldots, \frac{34}{2} = 17$ runs as follows:

$$\{1,5\},\{2,6\},\{3,7\},\{4,8\},\{9,13\},\{10,14\},\{11,15\},\{12,16\},\{17,21\},$$
$$\{18,22\},\{19,23\},\{20,24\},\{25,29\},\{26,30\},\{27,31\},\{28,32\},\{33,37\}.$$

Subtracting $i_1 = 1$ from the smallest element of each set A_i, we can define

$$d_{34} = 2^0+2^1+2^2+2^3+2^8+2^9+2^{10}+2^{11}+2^{16}+2^{17}+2^{18}+2^{19}+2^{24}+2^{25}+2^{26}+2^{27}+2^{32}.$$

It is immediate that $s_2(34\,d_{34}) = 34$ (we invite the reader to check that a_{34} is strictly smaller than d_{34}).

One can introduce a new restriction on Niven numbers in the following way: we define a *strongly q-Niven number* to be a q-Niven number whose base q digits are all 0 or 1. Obviously, every 2-Niven number is a strongly 2-Niven number. Other examples include

$$q + q^2 + \cdots + q^q, \qquad \text{or} \qquad q + q^3 + q^5 + \cdots + q^{2q+1},$$

which are both strongly q-Niven numbers for any base q. The related problem of investigating the statistical properties of the strongly q-Niven numbers seems interesting and we shall pursue this elsewhere.

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