Calculus I, Notes

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December 13th, 2007

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Preface

These lecture notes are written during the Fall semester of 2007 and the Spring semester 2008 for my students enroled in Calculus I. There are lots of calculus books out there which are very heavy. Yet, calculus concepts are just a few: the limit, the derivative and the definite integral. In these notes we would like to take an approach that goes to the matter of things. Most of the applications will be taken from the text we are using in class: [2]. The idea of using all transcendental functions from the start has nevertheless good pedagogical advantages. So, we are going to adopt the same perspective here although we would like to introduce all these functions later in a rigorous way by the use of the concept of definite integral. For instance, the usual definitions one needs to have are:

$$\ln x := \int_1^x \frac{1}{t} dt, x > 0 \quad \textit{and} \quad \arcsin x := \int_0^x \frac{1}{\sqrt{1-t^2}} dt, x \in [-1, 1],$$

and the rest of the properties of all the elementary functions follow from these definitions.

We begin with the concept of limits and introduce the so called fundamental limits. Exemplifying the concept of limit with nontrivial situations is not just a matter of taste but also choice that we make to show the connection with the derivatives of the elementary functions. Continuity is briefly studied and some applications of the Intermediate Value Theorem are given. This is mostly a prelude for the work needed with the definition of the derivative and the study of all differentiation rules. We then continue with usual applications such as related rates problems, implicit differentiation, Newton's approximation technique and the Mean Value Theorem and its corollaries. Finally the concept of Riemann integral and a few techniques of integration are given after the Fundamental Theorem of Calculus is discussed.

Chapter 1

Limits and The Main Elementary Functions

1.1 Fundamental Limits

> **Quotation:** *"The result of the mathematician's creative work is demonstrative reasoning, a proof, but the proof is discovered by plausible reasoning, by GUESSING" –George Polya, Mathematics and Plausible Reasoning, 1953.)*

The concept of limit is essential in the investigation of this mathematical subject called Calculus. The idea of limit can be intuitively given by some important examples.

Example 1: Let us consider the function

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

defined for all $x > 0$. Its graph is included in Figure 1.1.

From the graph of f we see that $f(x)$ gets closer and closer to a horizontal line, $y = 2.71...$, as x gets bigger and bigger; we formally say in a mathematical language that x goes or tends to infinity (symbol used for infinity is ∞). We assume this pattern continues as x grows indefinitely. This number that appears here magically is an important constant in mathematics and it is denoted by e in honor of the mathematician Leonhard Euler (1707-1783) who was first to coin the notation for this number.

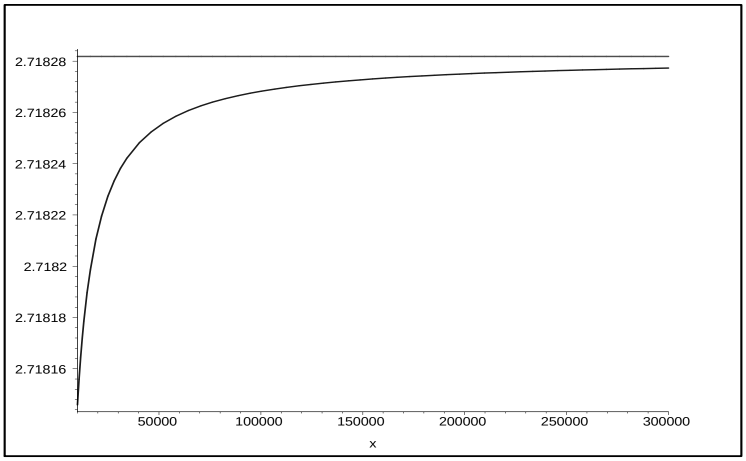

Figure 1.1: Plot of $f(x) = \left(1 + \frac{1}{x}\right)^x$, $x = 10000..300000$

This number is transcendental, i.e. there is no polynomial equation with integer coefficients that has e as its root, and its truncation to 20 decimals is

$$e \approx 2.71828182845904523536\ldots.$$

The fact about the behavior of the function f is recorded mathematically by writing

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.} \tag{1.1}$$

This is one of the fundamental limits that connects the behavior of polynomial functions with the exponential functions. In general the exponential functions are functions of the form $g(x) = a^x$ with $a \in (0,1) \cup (1,\infty)$. If a is the number e then the function is called the natural exponential function.

In the theory of limits for functions one can first introduce the limit of a particular type of functions which are called sequences. In general by *a sequence of real numbers* we just understand an infinite list a_1, a_2, ..., a_n, ... where a_k are real numbers. As one of the simplest examples is $a_n = \frac{1}{n}$. As n goes to ∞ then a_n gets closer and closer to zero. We write this like $\lim_{n\to\infty} \frac{1}{n} = 0$.

The precise meaning of the limit of a sequence is given in the following definition:

Definition 1.1.1. *We say that the number L is the limit of the sequences a_n if for every $\epsilon > 0$ there exists an index n (which depends on ϵ) such that $|a_m - L| < \epsilon$ for all $m \geq n$. A short and formal way to express that a_n has limit L (or a_n converges to L) is $\lim_{n\to\infty} a_n = L$.*

An equivalent way of writing (1.1) is

$$\lim_{a_n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e.$$

Definition 1.1.2. *In general, we say that a function f has limit L at $x = a$ if the sequence $f(a_n)$ converges to L for every sequence a_n convergent to a.*

We are going to prove (1.1) later on in the course after the formal definition of exponential functions by use of definite integrals has been introduced. At this point we are just going to take (1.1) for granted. To avoid circular reasoning we have to avoid using (1.1) as an important fact in the process of defining the exponential function and of course all of its properties that lead to this fundamental limit.

At this point we would like to derive some other elementary limits using properties of limits and these fundamental limits.

A list of the basic properties of limits of sequences or functions which can be derived from the definitions of limits is given here:

1. $\lim_{x\to a} \mathit{constant} = \mathit{constant}$
2. $\lim_{x\to a} \mathit{constant} f(x) = \mathit{constant} \lim_{x\to a} f(x)$, $\lim_{n\to\infty} \mathit{constant} a_n = \mathit{constant} \lim_{n\to\infty} a_n$
3. $\lim_{x\to a} f(x) \pm g(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
4. $\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$
5. $\lim_{x\to a} f(x)/g(x) = (\lim_{x\to a} f(x))/(\lim_{x\to a} g(x))$
6. $\lim_{x\to a} f(x)^r = (\lim_{x\to a} f(x))^r$

Let us work out an example in which these properties are used.

Example: Compute $\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x$.

Since

$$\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right]^3$$

using Property 6, and the substitution $\frac{x}{3}=t$ $(t\to\infty)$ we get

$$\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x = \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right]^3 = e^3.$$

The Property 6 above can be extended to lots of other elementary functions. What do we understand by elementary functions? First let us start with the basic elementary functions:

1. Polynomials: $p(x) = c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$, $Domain = \mathbb{R}$;
2. Power Functions: $g(x) = x^r$, $Domain = (0,\infty)$, $r \in \mathbb{R}$;
3. Trigonometric Functions: *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*;
4. Exponential Functions: $h(x) = a^x$, $x \in \mathbb{R}$;
5. Logarithmic Functions: $i(x) = \log_a(x)$, $Domain = (0,\infty)$;
6. Inverse Trigonometric Functions: *arcsin*, *arccos*, *arctan*, *arcsec*;

All the above classes of functions except polynomials are called transcendental functions. Their precise definitions will be given later hence the reason for the title of the text book "Early Transcendentals".

An elementary function is then the result of a class of functions which is closed under composition of functions and the usual operations with functions such as addition multiplication or division. Let us give a few examples:

$$j(x) = \left[\log_2(x^3+2x)+\sin(x)\right]^{\frac{2}{e^x}}, \quad k(x) = \frac{\arcsin(2^x+3^{2x})}{\arctan(x)+\ln(2x+1)},$$

or all the hyperbolic functions and their inverses.

Such functions may have complicated domains but whatever these domains are they will play an important role in what follows. The Property 6 can be extended (shown to hold true) to a function as above, say F, in the following way:

$$\lim_{x\to a} F(f(x)) = F(\lim_{x\to a} f(x)), \tag{1.2}$$

whenever $\lim_{x\to a} f(x)$ is in the domain of F and the composition $F(f(x))$ makes sense. The reason for which (1.2) happens is in fact a more general (at least formally) property:

$$\lim_{x\to b} F(x) = F(b),\ \ b \in Domain(F), \tag{1.3}$$

which is called continuity of F at the point b. In other words we have the following theorem:

Theorem 1.1.3. *Every elementary function is continuous at each point in its domain of definition.*

As an application of this theorem let us derive another fundamental limit which is equivalent to (1.1) and it is an intimate connection between polynomials and the natural logarithmic function:

$$\boxed{\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.} \tag{1.4}$$

Since ln is continuous at the point e we obtain

$$\lim_{x\to\infty} \ln(1+\frac{1}{x})^x = \ln e = 1,\ \ or \lim_{x\to\infty} x\ln(1+\frac{1}{x}) = 1,$$

and if we substitute $y = \frac{1}{x} \to 0$ we get $\lim_{y\to 0} \frac{\ln(1+y)}{y} = 1$ which is nothing else but (1.4). Of course, if one assumes (1.4) the first fundamental limit, (1.1), follows.

The third fundamental limit can be derived from (1.4) and it intimately connects the polynomials with the exponentials:

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1.} \tag{1.5}$$

Indeed if we set $y = \ln(1+x) \to 0$ as $x \to 0$ (continuity of ln at the point 1) we obtain $x = e^y - 1$ and so (1.4) becomes $\lim_{y\to 0} \frac{y}{e^y - 1} = 1$. Taking reciprocals we get (1.5).

Next let us derive the fourth fundamental limit which intimately connects the polynomials with the power functions:

$$\boxed{\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha, \quad \alpha \in \mathbb{R}.} \tag{1.6}$$

Using the fact that the logarithmic function is the inverse of the exponential function, i.e. $a = e^{\ln a}$, we have

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \lim_{x\to 0} \frac{e^{\ln(1+x)^\alpha} - 1}{x} = \lim_{x\to 0} \frac{e^{\alpha\ln(1+x)} - 1}{\alpha\ln(1+x)} \frac{\alpha\ln(1+x)}{x}.$$

Because $t = \alpha\ln(1+x) \to 0$ as $x \to 0$ and using (1.4) and (1.5) we obtain

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \lim_{y\to 0} \frac{e^y - 1}{y} \lim_{x\to 0} \frac{\alpha\ln(1+x)}{x} = \alpha.$$

Let us work an exercise in which (1.6) plays an important role.

Exercise: Calculate the limit $\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$.

Solution: Changing the variable of the limit to $y = x - 1$ we see that while $x \to 1$ then $y \to 0$. Hence the limit becomes

$$\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{x - 1} = \lim_{y\to 0} \frac{(1+y)^{1/3} - 1}{y} = \frac{1}{3}.$$

The fifth fundamental limit which cannot be derived from the previous ones is one that intimately connects the polynomials with trigonometric functions:

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.} \tag{1.7}$$

We are going to show this property when the trigonometric functions will be defined rigorously with the concept of definite integral.

1.2 Continuity and piecewise functions

We have seen in Theorem 1.1.3 that every elementary functions is continuous on its domain of definition.

A new class of functions which appears often in applications we will refer to it here as *piecewise functions*. This set of functions is important also within mathematics as a theoretical tool since it provides a good pool for examples and counterexamples.

Let us consider such an example:

$$f(x) = \begin{cases} \frac{\sin 2x}{x} \quad if\ x > 0, \\ \\ 2 \quad if\ x = 0, \\ \\ \frac{\ln(1+2x)-1}{x} \quad if\ x < 0 \end{cases} .$$

This function is continuous at every point different of zero since the rules for each branch are elementary functions well defined on those intervals. At $x = 0$ we have

$$\lim_{x\to 0^+} \frac{\ln(1+2x)-1}{x} = 2\lim_{x\to 0} \frac{\ln(1+2x)-1}{2x} = 2$$

and

$$\lim_{x\to 0^-} \frac{\sin 2x}{x} = 2\lim_{x\to 0} \frac{\sin 2x}{2x} = 2.$$

Hence we conclude that $\lim_{x\to 0} f(x) = 2 = f(0)$ and so this function is continuous.

On the other hand if we simply change the definition of f to

$$\hat{f}(x) = \begin{cases} \frac{\sin 2x}{x} \quad if\ x > 0, \\ \\ 3 \quad if\ x = 0, \\ \\ \frac{\ln(1+2x)-1}{x} \quad if\ x < 0 \end{cases} .$$

In this case clearly $\hat{f}$ is not continuous, we say it is *discontinuous*, and since the limit exists at this point we call such a point a *removable discontinuity*.

A more interesting example is the following function which does not have a limit at zero:

$$g(x) = \begin{cases} \sin(\frac{1}{x}) \ \ if \ x > 0, \\ \\ 2 \ \ if \ x = 0, \\ \\ x\sin(\frac{1}{x}) \ \ if \ x < 0 \end{cases},$$

although the left hand side limit exists since $|x\sin(\frac{1}{x})| \leq |x| \to 0$. This forces $\lim_{x\to 0^-} g(x) = 0$. This principle is known as the *squeeze theorem.* So, g is discontinuous at $x = 0$ and such a discontinuity is called an *essential discontinuity.*

An important theorem that is used often in mathematics is the Intermediate Value Theorem:

Theorem 1.2.1. *Consider a continuous function on a closed interval* $[a, b]$ *and a number* c *between* $f(a)$ *and* $f(b)$*. Then there exists a value* $x \in [a, b]$ *such that* $f(x) = c$*.*

The proof of this theorem is beyond the scope of the course so we invite the interested students read a proof of it from a real analysis textbook.

As an application let us work the following problem:

If a *and* b *are positive numbers, prove that the equation*

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0 \tag{1.8}$$

has at least one solution in the interval $(-1, 1)$*.*

The equation is equivalent to $a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. So, if we denote by $p(x) = a(x^3 + x - 2) + b(x^3 + 2x - 1)$ we notice that, p is continuous on $[-1, 1]$ and $p(-1) = -4a < 0$ and $p(1) = 2b > 0$. Hence 0 is in between the two values of p at the endpoints of the interval $[-1, 1]$ and so, by the Intermediate Value Theorem, there must be a $c \in (-1, 1)$ such that $p(c) = 0$. This means c is a solution of the original equation.

A related problem and a more precise statement about the possible zeroes of (1.8) will be two show that the equation (1.8) has at least one solution in the interval $(\alpha, 1)$ where $\alpha = \frac{\sqrt{5}-1}{2} \approx 0.6180$ (reciprocal of the so called golden ratio number).

Indeed, the polynomial above can be written in the form $p(x) = a(x-1)(x^2 + x + 2) + b(x+1)(x^2 + x - 1)$ and α is a root of the polynomial $x^2 + x - 1$. Hence $p(\alpha) = a(\alpha - 1)3 < 0$ and $p(1) = 2b > 0$. Therefore the same argument applies for the interval $(\alpha, 1)$.

Chapter 2

Derivatives and the rules of differentiation

2.1 Derivatives of polynomials

> **Quotation:** *A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. –George Polya*

The concept of differentiation is nevertheless the most important in calculus. We are go to start with the geometric question that leads to this notion. Consider one of the important curves that one plays in geometry: the circle. Taking a point on this circle one can draw several lines passing through this point but only one will intersect the circle at only that particular point. We usually call this line the tangent line to the circle at the given point. We know that such a line can be obtained by just taking the perpendicular to the corresponding radius of the point where the tangent is to be drawn.

What if we have some other types of curves? How do we first even define the concept of tangent line and how do we compute it?

Let us start with the curve of equation $y = f(x)$ and suppose we take $P = (a, f(a))$ a point on this curve. For another point close to P, say $Q = (x, f(x))$ we can calculate the slope of the secant line $\overline{PQ}$:

$$\frac{f(x) - f(a)}{x - a}.$$

Intuitively, when $x \to a$, this slope tends to have the limiting value of the "tangent" line to the curve at this point. This is actually what we will take by definition to be the tangent line at $(a, f(a))$ to $y = f(x)$:

$$\boxed{y - f(a) = f'(a)(x - a)}$$

where $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ if this limit exists. We call this limit the derivative of f at a. Other notations used for this limit are: $\frac{df}{dx}(a)$ or $\frac{df}{dx}|_{x=a}$. We can look at this calculation as a function too if we define f' (the derivative of f) as being

$$\boxed{f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}} \tag{2.1}$$

for all $x \in \mathit{Domain}(f') := \{x| \textit{ all real } x \textit{ where the limit } (2.1) \textit{ exists}\}$.

We can say that calculus is the study of the operation $f \to f'$ as applied mainly to elementary functions. There are quite a few surprises and interesting stories about this "simple" transformation.

One of the beginning stories is that each of the fundamental limits, that we have identified in Chapter I, represents the derivative of one of the basic elementary functions at a certain point and it is enough to calculate the derivative in general.

Let us start with the derivative of a power function:

$$\alpha = \lim_{t\to 0} \frac{(1+t)^\alpha - 1}{t} = \lim_{x\to 1} \frac{x^\alpha - 1}{x - 1} = f'(1)$$

where $f(x) = x^\alpha$, $x > 0$.

Let us calculate the derivative at any other point $a > 0$:

$$f'(a) = \lim_{x\to a} \frac{x^\alpha - a^\alpha}{x - a} = \lim_{x\to a} \frac{a^\alpha((\frac{x}{a})^\alpha - 1)}{a(\frac{x}{a} - 1)} = a^{\alpha-1} \lim_{t\to 1} \frac{t^\alpha - 1}{t - 1} = \alpha a^{\alpha-1}.$$

Hence, we have the derivative of a power function, also known as the power rule:

$$\boxed{\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1},\ x > 0.}$$

Next, let us derive the derivative of the exponential function.

Consider $g(x) = e^x$ and $a \in \mathbb{R}$ arbitrary. Then

$$g'(a) = \lim_{t \to a} \frac{g(t) - g(a)}{t - a} = \lim_{t \to a} \frac{e^t - e^a}{t - a} = \lim_{t \to a} \frac{e^a(e^{t-a} - 1)}{t - a}$$

and after the substitution $t - a = x$, since $x \to 0$ we obtain

$$g'(a) = e^a \lim_{x \to 0} \frac{e^x - 1}{x} = e^a.$$

Therefore we have $\boxed{\frac{d}{dx} e^x = e^x, x \in \mathbb{R}.}$

But what if we have a simple change in the base of the exponential function? Say, $g(x) = b^x$ with $b > 0$ and $b \neq 1$.

Then

$$g'(a) = \lim_{t \to a} \frac{g(t) - g(a)}{t - a} = \lim_{t \to a} \frac{b^t - b^a}{t - a} = \lim_{t \to a} \frac{b^a(b^{t-a} - 1)}{t - a} =$$

$$b^a \lim_{x \to 0} \frac{b^x - 1}{x} = b^a \lim_{x \to 0} \frac{e^{x \ln b} - 1}{x \ln b} \ln b = b^a \ln b.$$

Hence, $\boxed{\frac{d}{dx} b^x = b^x \ln b, x \in \mathbb{R}.}$

We will derive finally the derivative of the most common trigonometric function: $h(x) = \sin x$ defined for all radian angles $x \in \mathbb{R}$.

For fixed $a \in \mathbb{R}$ we have

$$h'(a) = \lim_{t \to a} \frac{\sin t - \sin a}{t - a} = \lim_{t \to a} \frac{2 \sin \frac{t-a}{2} \cos \frac{t+a}{2}}{t - a}$$

using the formula from trigonometry $\sin \alpha - \cos \beta = 2 \sin \frac{\alpha - \beta}{2} \cos \frac{\alpha + \beta}{2}$. Then we change the variable $\frac{t-a}{2} = x$ and notice that $x \to 0$ as $t \to a$. That gives

$$h'(a) = \lim_{x \to 0} \frac{\sin x}{x} \cos(x + a) = \cos a,$$

and so $\boxed{\frac{d}{dx} \sin x = \cos x, x \in \mathbb{R}.}$

For the cosine we can can do a similar calculation. Let $i(x) = \cos x$ with $x \in \mathbb{R}$.

Bibliography

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