# ON POWER BOUNDED OPERATORS

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Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called *power bounded* (notation: $T \in \mathcal{PW}(\mathcal{H})$) if there exists a constant $M(\geq 1)$ such that

$$\|T^n\| \leq M, \qquad n \in \mathbb{N}, \tag{1}$$

and $T$ is called *polynomially bounded* (notation: $T \in \mathcal{PB}(\mathcal{H})$) if there exists a constant $M(\geq 1)$ such that

$$\|p(T)\| \leq M\|p\|_\infty \tag{2}$$

for every polynomial $p$, where $\|p\|_\infty = \sup\{|p(z)| : z \in \mathbb{C}, \ |z| \leq 1\}$. The smallest number $M$ satisfying (1) (resp., (2)) is called the *power bound* ( resp., the *polynomial bound*) of $T$ and will be denoted by $M_w(T)$ (resp., $M_p(T)$), or simply $M_w$ (resp., $M_p$) when no confusion is possible. One knows (cf. [1], [2], [4]) that $\mathcal{PW}(\mathcal{H})$ strictly contains the class $\mathcal{PB}(\mathcal{H})$, but there is a theorem of Nagy [3] which says that every $T \in \mathcal{PW}(\mathcal{H})$ such that $T^{-1}$ exists and belongs to $\mathcal{PW}(\mathcal{H})$ is similar to a unitary operator, and therefore is polynomialy bounded. The purpose of this note is to establish the following two stronger results than the above-mentioned consequence of Nagy's theorem.

**Theorem 1.1.** *Suppose $T \in P\mathcal{W}(\mathcal{H})$(with $M_w(T) > 1$) and the following inequality holds for some positive number $\alpha$ and a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ :*

$$1/n_k \sum_{j=0}^{n_k} T^{*j}T^j \geq \alpha(I - P_{ker(T)}), \tag{3}$$

*where $P_{ker(T)}$ is the (orthogonal) projection on the kernel of $T$. Then $T \in \mathcal{PB}(\mathcal{H})$ and the polynomial bound $M_p$ of $T$ satisfies*

$$M_p \leq M_w(T)^3 \left(\frac{M_w(T)^2-1}{\alpha ln M_w(T)}\right)^{1/2} + 1. \tag{4}$$

**Theorem 1.2.** *Suppose $T \in \mathcal{PW}(\mathcal{H})$ and the following inequality holds for some positive number $\alpha$ and a strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers converging to 1 :*

$$(1-t_k)\sum_{j=0}^{\infty} t_k^j T^{*j}T^j \geq \alpha(I - P_{ker(T)}). \tag{5}$$

*Then $T \in \mathcal{PB}(\mathcal{H})$, and the polynomial bound $M_p$ of $T$ satisfies*

$$M_p \quad \leq \quad \left(\frac{14}{\alpha}\right)^{1/2} M_w^3. \tag{6}$$

As mentioned above, the following is an immediate consequence of either Theorem 1.1 or Theorem 1.2 .

**Corollary 1.3 (Nagy [4]).** *If $T \in \mathcal{PW}(\mathcal{H})$ is invertible and $T^{-1} \in \mathcal{PW}(\mathcal{H})$, then $T$ is polynomially bounded .*

In order to prove Theorem 1.1, we use the following lemma, which is well known [5].

**Lemma 1.4.** *Suppose $S \in \mathcal{L}(\mathcal{H})$ is such that $S^m$ is a contraction for some integer $m \geq 2$. Then $S$ is similar to a contraction, and, in particular,*

$$A = (I + S^*S + ... + S^{*(m-1)}S^{(m-1)})^{1/2} \tag{7}$$

*is an invertible operator that satisfies*

$$||ASA^{-1}|| \leq 1. \tag{8}$$

**Proof.** Clearly $A$ is an invertible selfadjoint operator. To establish (8) it is enough to check that

$$||ASA^{-1}h|| \leq ||h|| \quad , \quad h \in \mathcal{H}. \tag{9}$$

For a given h, define $g = A^{-1}h$, and hence (9) becomes equivalent to

$$< A^2 Sg, Sg > \ \leq \ < A^2 g, g > . \tag{10}$$

Using (7), we see that (10) is equivalent to

$$\sum_{j=1}^{m} \|S^j g\|^2 \leq \sum_{j=0}^{m-1} \|S^j g\|^2,$$

which is true since $\|S^m g\| \leq \|g\|$. ■

**Proof of Theorem 1.1.** For brevity we write $M = M_w(T) > 1$. For each $n \in \mathbb{N}$, set $\alpha_n = M^{1/n}$, $\beta_n = \alpha_n^{-1}$, and note that $\beta_n < 1 < \alpha_n$. Since $\|(\beta_n T)^n\| \leq 1$ for each $n \in \mathbb{N}$, we may apply Lemma 1.4 to obtain for each $n \in \mathbb{N}$ a contraction $C_n$ such that

$$\beta_n T = A_n^{-1} C_n A_n \ , \tag{11}$$

where $A_n = (\sum_{j=0}^{n-1} \beta_n^{2j} T^{*j} T^j)^{1/2}$. Consider now an arbitrary polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_l z^l$. Then

$$p(T) = p(\alpha_n A_n^{-1} C_n A_n) = A_n^{-1} p(\alpha_n C_n) A_n \ . \tag{12}$$

Applying the von Neumann inequality to $C_n$ and the polynomial $q_n(z) = p(\alpha_n z)$, we conclude from (12) that

$$\|p(T)\| \leq \|A_n^{-1}\| \|A_n\| \|q_n\|_\infty, \quad n \in \mathbb{N}. \tag{13}$$

Let us observe now that for each $n \in \mathbb{N}$,

$$\|A_n\|^2 = \|A_n^2\| \leq \sum_{j=0}^{n-1} \beta_n^{2j} M^2 = M^2(1-\beta_n^{2n})/(1-\beta_n^2) = M^2(1-M^{-2})/(1-\beta_n^2),$$

so

$$\|A_n\| \leq (M^2-1)^{1/2}/(1-\beta_n^2)^{1/2} \ . \tag{14}$$

Moreover, for each $n \in \mathbb{N}$, $\|A_n^{-1}\| = \gamma(A_n)^{-1}$, where $\gamma(A_n)$ is the greatest number $\gamma > 0$ with the property that $\|A_n h\| \geq \gamma \|h\|$ for all $h \in \mathcal{H}$. Equivalently,

$$< A_n^2 h, h > \geq \gamma(A_n)^2 < h, h > \ , \ \ h \in \mathcal{H}, \quad n \in \mathbb{N}.$$

Consider now the case that $ker(T) = \{0\}$. Let $\{n_k\}$ be the sequence from (3). Then

$$A_{n_k}^2 = \sum_{j=0}^{n_k-1} \beta_{n_k}^{2j} T^{*j} T^j \geq \beta_{n_k}^{2n_k} \sum_{j=0}^{n_k-1} T^{*j} T^j \geq \beta_{n_k}^{2n_k} n_k \alpha I = n_k M^{-2} \alpha I, \quad k \in \mathbb{N}.$$

Therefore, $\gamma(A_{n_k}) \geq n_k^{1/2} M^{-1} \alpha^{1/2}$ for each $k \in \mathbb{N}$, which implies that

$$\|A_{n_k}^{-1}\| = \gamma(A_{n_k})^{-1} \leq n_k^{-1/2} M \alpha^{-1/2}. \tag{15}$$

Thus, from (14) and (15) we get

$$\|A_{n_k}\| \|A_{n_k}^{-1}\| \leq M(M^2-1)^{1/2} \alpha^{-1/2} / (n_k(1-\beta_{n_k}^2))^{1/2}\ ,\ k \in \mathbb{N}. \tag{16}$$

A simple continuity argument shows that for $p$ fixed we have

$$lim_{k\to\infty} \|q_{n_k}\|_\infty = \|p\|_\infty. \tag{17}$$

Going back to (13), and taking into account (16) and (17), we can let k go to infinity and obtain the inequality

$$\|p(T)\| \leq M((M^2-1)/2lnM)^{1/2} \alpha^{-1/2} \|p\|_\infty \tag{18}$$

by using the formula (from elementary calculus)

$$lim_{n\to\infty}(M^{2/n}-1)n = 2ln\ M\ .$$

Thus, in this case, $T \in \mathcal{PB}(\mathcal{H})$ and (4) is valid.

Let's consider now the general case. With respect to the decomposition $\mathcal{H} = (ker\ T) \oplus (ker\ T)^\perp$, $T$ has an operator matrix

$$T = \begin{bmatrix} 0 & S \\ & \\ 0 & Q \end{bmatrix} \tag{19}$$

where $S : (ker\ T)^\perp \to (ker\ T)$ is a bounded linear operator, $Q \in \mathcal{PW}((ker\ T)^\perp)$, and $M_w(Q) \leq M_w(T)$. For each polynomial $p$ one sees easily that

$$p(T) = \begin{bmatrix} p(0)I & Sq(Q) \\ & \\ 0 & p(Q) \end{bmatrix}$$

where $q(z) = (p(z) - p(0))/z$. Therefore, since $\|q\|_\infty \leq 2\|p\|_\infty$, it is sufficient to show that $Q$ is polynomially bounded and has an appropriate polynomial bound. We want to use the first case, so let us observe that

$$(20) \qquad T^{*k}T^k \leq \begin{bmatrix} 0 & 0 \\ & \\ 0 & (\|S\|^2 + \|Q\|^2)Q^{*k-1}Q^{k-1} \end{bmatrix}, \quad k \in \mathbb{N}.$$

But (3) and (20) together yield

$$(\|S\|^2 + \|Q\|^2)/(n_k - 1) \sum_{j=0}^{n_k - 1} Q^{*j}Q^j \geq (\alpha - (\alpha + 1)/(n_k - 1))I_{(ker\ T)^\perp}.$$

In particular this says that if $h \in ker(Q) \cap (ker\ T)^\perp$, then

$$(\|S\|^2 + \|Q\|^2)/(n_k - 1) < h, h > \ \geq \ (\alpha - (\alpha + 1)/(n_k - 1)) < h, h >,$$

and leting k go to infinity we obtain that $h = 0$. Hence, Q satisfies the condition (3) in the case when $ker(Q) = \{0\}$ for $\alpha' = (\alpha - \epsilon)/(\|S\|^2 + \|Q\|^2) > 0$ and a subsequence $\{n_k - 1\}$ for k large enough (depending upon $\epsilon$). Therefore, we obtain from the previous case,

$$(21) \qquad \|p(Q)\| \leq M((M^2 - 1)/2lnM)^{1/2}\alpha^{-1/2}(\|S\|^2 + \|Q\|^2)^{1/2}\|p\|_\infty ,$$

since $\epsilon > 0$ was arbitrary. Finally we get

$$\|p(T)\| \leq (M((M^2 - 1)/2lnM)^{1/2}\alpha^{-1/2}(\|S\|^2 + \|Q\|^2)^{1/2}\|S\| + 1)\|p\|_\infty \leq (M^3((M^2 - 1)/lnM)^{1/2}\alpha^{-1/2} + 1)\|p\|_\infty,$$

which is what we wanted to show. ■

We want to consider now the continuous analog of Theorem 1.1.

**Proof of Theorem 1.2.** Let us define for $T \in \mathcal{PW}(\mathcal{H})$ and every $t \in [0, 1)$ the self-adjoint invertible operator

$$(22) \qquad A_t = (1 - t)^{1/2}(\sum_{j=0}^{\infty} t^j T^{*j} T^j)^{1/2}.$$

First, observe that this operator is well-defined for $T \in \mathcal{PW}(\mathcal{H})$, and moreover

$$\|A_t\|^2 = \|A_t^2\| \leq (1 - t)\sum_{j=0}^{\infty} t^j \|T^j\|\|T^{*j}\| \leq M_w(T)^2.$$

As before, let us consider the case when $ker(\,T) = \{0\}$. If (5) is satisfied, then $\|A_t^{-1}\| \leq \alpha^{-1/2}$ at least for $t = t_k$.
Now observe that for any $h \in \mathcal{H}$ we have

$$(1-t)\|A_t^{-1}h\|^2 + t\|A_t T A_t^{-1}h\|^2 = \|h\|^2,$$

which, in particular, says that $t^{1/2}A_t T A_t^{-1}$ is a contraction. Hence we can use the idea from the proof of the Theorem 1.1 to get that

$$\|p(T)\| \leq \|A_{t_k}\|\|A_{t_k}^{-1}\|\|q_k\|_\infty, \qquad k \in \mathbb{N},$$

where $q_k(z) = p(t_k^{-1/2}z)$ for given any polynomial $p$. Letting k go to infinity we get the inequality

$$\|p(T)\| \leq M_w(T)\alpha^{-1/2}\|p\|_\infty,$$

which is what we wanted to show in the case $ker(T) = 0$. In the general case, if $T$ has the decomposition (19), by using the inequality (20) and the hypothesis (5), we have that

$$(\|S\|^2 + \|Q\|^2)(1-t_k)\sum_{j=1}^{\infty} t_k^j Q^{*j-1}Q^{j-1} \geq (\alpha - 1 + t_k)I_{(ker\ T)^\perp},$$

which says, first, that $ker(Q) = 0$ and thus that $Q$ is as in the first case. Therefore. we finally get

$$\|p(T)\| \leq M_w(T)\left\{(3+4\|S\|^2)\left(\frac{\|S\|^2+\|Q\|^2}{\alpha}\right)\right\}^{1/2}\|p\|_\infty \leq$$
$$\leq \left(\frac{14}{\alpha}\right)^{1/2} M_w(T)^3\|p\|_\infty,$$

which was to be proved. ∎

An easy corollary of Theorem 1.2 is the following generalization.

**Corollary 1.5.** *Suppose $T \in \mathcal{PW}(\mathcal{H})$ and the following inequality holds for some $n \in \mathbb{N}$, some positive number $\alpha$, and a strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers converging to* $1$ :

$$(1-t_k)\sum_{j=0}^{\infty} t_k^j T^{*j}T^j \geq \alpha(I - P_{ker(T^n)}). \tag{23}$$

*Then $T$ is polynomially bounded.*

**Proof.** With respect to the decomposition $\mathcal{H} = ker(T^n) \oplus (ker(T^n))^{\perp}$, $T$ has the operator matrix

$$T = \begin{bmatrix} A & B \\ & \\ C & D \end{bmatrix}.$$

Since $ker(T^n)$ is an invariant subspace for $T$, the operator C must be zero. In addition, we have that

$$T^n = \begin{bmatrix} 0 & E \\ & \\ 0 & F \end{bmatrix},$$

where

$$A^n = 0, \quad F = D^n, \quad E = \sum_{j=0}^{j=n} A^j B D^{n-j}. \tag{24}$$

Now, for an arbitrary operator $T$, $T \in \mathcal{PB}(\mathcal{H})$ if and only if $T^m \in \mathcal{PB}(\mathcal{H})$ for some $m \in \mathbb{N}$. This can be easily seen if we observe that for any polynomial $p$, there exists a unique decomposition of the form

$$p(z) = p_1(z) + zp_2(z) + z^2p_3(z) + ... + z^{m-1}p_m(z),$$

where $p_1,\ p_2,\ p_3, ...,\ p_m$ are polynomials in $z^m$ and $\|p_j\|_\infty \leq \|p\|_\infty$ for $j = 1,\ 2, ...,\ m$. Hence, it suffices to show that $F$ and therefore $D$ is polynomially bounded. But now, since for any integer $k \geq 0$ we have

$$(I - P_{ker(T^n)})T^{*k}T^k(I - P_{ker(T^n)}) = \begin{bmatrix} 0 & 0 \\ & \\ 0 & D^{*k}D^k \end{bmatrix},$$

it follows from (23) multiplying from the left and form the right by $I - P_{ker(T^n)}$, we obtain that $D$ satisfies the hypothesis of the Theorem 1.1. This means that $D$ is polynomialy bounded, so F and T are also. ∎

**Comments.** If we start with a contraction $T$, let us show that the function $A_t$ defined in (22) satisfies $A_t \geq A_s$ for $0 \leq t < s \leq 1$. Indeed, since $A^2 \geq B^2$ for positive simidefinite operators implies $A \geq B$, it is enough to check that $A_t^2 \geq A_s^2$ $(t < s)$. This is equivalent to

$$(1-t)\sum_{j=0}^{\infty} t^j \|T^j h\|^2 \geq (1-s)\sum_{j=0}^{\infty} s^j \|T^j h\|^2 \ , \ h \in \mathcal{H} \ ,$$

and this can be written in the following equivalent form which is clearly true for $T$ a contraction :

$$(s-t)(\|h\|^2 - \|Th\|^2) + (s^2-t^2)(\|Th\|^2 - \|T^2h\|^2) +$$
$$(s^3-t^3)(\|T^2h\|^2 - \|T^3h\|^2) + (s^4-t^4)(\|T^3h\|^2 - \|T^4h\|^2) + ... \geq 0 \ \ .$$

Therefore, it is interesting to ask: what is the class of operators for which the function $t \to A_t$ is decreasing on some interval $(\beta, 1)$, $\beta > 0$? In this connection one can easily prove, using ideas similar to those above, the following.

**Theorem 1.6.** *Suppose $T \in \mathcal{PW}(\mathcal{H})$, the positive-operator-valued function*

$$t \to (1-t)^{1/2}(\sum_{j=0}^{\infty} t^j T^{*j} T^j)^{1/2}, \quad t \in [0,1)$$

*is decreasing on some interval $(\beta, 1)$, $\beta > 0$ , and the inequalilty (5) holds for some positive number $\alpha$ and a strictly increasing sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers converging to 1. Then $T$ is similar to a contraction.*

Another natural question is weather we can weaken the assumption (23) to

$$(1-t_k)\sum_{j=0}^{\infty} t_k^j T^{*j} T^j \geq \alpha(I - P_{\overline{\cup_{n=0}^{\infty} ker(T^n)}}), \tag{25 }$$

and preserve the conclusion in Corollary 1.5. The same counterexample of Foguel in [2] shows that there exists an operator satisfying (25) which is not polynomially bounded.

**Remark.** This paper constitutes part of the author's Ph.D. thesis written at Texas A&M University under the direction of Professor Carl Pearcy. We thank the referee of this paper who suggested the idea for improving Corollary 1.5.

## References

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