ON WAVELET SETS

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ABSTRACT: It is proved that associated with every wavelet set is a closely related "regularized" wavelet set which has very nice properties. Then it is shown that for many (and perhaps all) pairs E, F of wavelet sets the corresponding MSF wavelets can be connected by a continuous path in $L^2(\mathbb{R})$ of MSF wavelets for which the Fourier transform has support contained in $E \cup F$. Our technique applies, in particular, to the Shannon and Journe wavelet sets.

1. Introduction.

In [2] the notion of a wavelet set was introduced, and the question was raised whether the set of all one-dimensional wavelets is connected (in the norm topology on $L^2(\mathbb{R})$). Wavelet sets were also introduced independently and simultaneously as the support sets of MSF (Minimally Supported Frequency) wavelets in the sequence of papers [4], [8], and [9], in which the connectivity problem was raised. (See also the recent excellent book [7].) The problem has been treated by several additional authors (cf. [3], [10], [6]) but is not solved in complete generality. The purpose of this note is to introduce a different approach to the connectivity problem which seems to be very natural and simple and proves somewhat more in several particular cases.

We begin by introducing some preliminary terminology and notation. The measurable space under consideration will always be $\mathbb{R}$ together with the σ-ring $\mathbb{L}$ of Lebesgue measurable sets, and Lebesgue measure on this space will be denoted by μ. The L^2-space with respect to μ will be written, as above, simply as $L^2(\mathbb{R})$. Throughout the paper, we shall need the equivalence relation on $\mathbb{L}$ defined by $F \sim G$ if $F \bigtriangledown G$ is a null set $[\mu]$. Following roughly the ideas in [2], we redefine some notions and establish the connections between them. Recall (cf. [2]) that a function $w \in L^2(\mathbb{R})$ is a wavelet if the family of functions $\{w_{j,k}\}_{j,k\in\mathbb{Z}}$ defined by

$$w_{j,k}(s) = 2^{j/2} w(2^j s + k), \quad s \in \mathbb{R},\ j, k \in \mathbb{Z}, \tag{1}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

We say that a subset G of $\mathbb{R}$ with positive measure is a *wavelet set* if $\frac{1}{\sqrt{\mu(G)}}\chi_G = \mathcal{F}(w)$, where w is a wavelet in $L^2(\mathbb{R})$ and $\mathcal{F}$ is the Fourier-Plancherel transform on $L^2(\mathbb{R})$. A measurable subset

G of $\mathbb{R}$ is called a *2-dilation generator of a partition* of $\mathbb{R}$ if the sets

$$2^k G := \{2^k s : s \in G\},\ \ k \in \mathbb{Z},$$

are disjoint and $\bigcup_{k\in\mathbb{Z}} 2^k G \sim \mathbb{R}$, and G is a 2π*-translation generator of a partition* of $\mathbb{R}$ if the sets

$$G + 2k\pi := \{s + 2k\pi : s \in G\},\ \ k \in \mathbb{Z},$$

are disjoint and $\bigcup_{k\in\mathbb{Z}}(G+2k\pi) \sim \mathbb{R}$. A measurable subset G of $\mathbb{R}$ is *translation congruent modulo* 2π to a (measurable) set H if there exists a measurable bijection $\varphi : G \to \varphi(G)$ such that $\varphi(s) - s$ is an integral multiple of 2π for every s in G and $\varphi(G) \sim H$. Analogously, G is said to be *dilation congruent modulo* 2 to a (measurable) set H if there exists a measurable bijection $\psi : G \to \psi(G)$ such that $\psi(s)/s$ is an integral power of 2 for every s in G and $\psi(G) \sim H$.

Lemma 1.1. *The following conditions are equivalent for any measurable subset G of $\mathbb{R}$:*

(*a*) *G is a wavelet set,*

(*b*) *there exists a set $G' \in \mathbb{L}$ such that $G \sim G'$ and G' is a 2-dilation generator of a partition of* $\mathbb{R}$ *and a* 2π*-translation generator of another partition of* $\mathbb{R}$*,*

(*c*) *the Littlewood-Paley wavelet set $E = [-2\pi, -\pi) \cup [\pi, 2\pi)$ is translation congruent modulo* 2π *and dilation congruent modulo 2 to G.*

Proof. Assume that G is a wavelet set. Then, by definition, there exists a wavelet w such that $\frac{1}{\sqrt{\mu(G)}}\chi_G = \mathcal{F}(w)$. Thus the family of vectors $\{w_{j,k}\}_{j,k\in\mathbb{Z}}$ defined in (1) is an orthonormal basis of $L^2(\mathbb{R})$. Since $\mathcal{F}$ is a unitary transformation and $\mathcal{F}(w_{j,0}) = \frac{1}{\sqrt{\mu(2^j G)}}\chi_{2^j G}$ for all j in $\mathbb{Z}$, it follows that all of the sets $2^j G \cap 2^k G$, $j \neq k$ $(j,k \in \mathbb{Z})$, are null sets. We can clearly find a measurable subset G_1 of G such that $G_1 \sim G$ and the family $\{2^j G_1\}_{j\in\mathbb{Z}}$ is disjoint. Since for all $j,\ k \in \mathbb{Z}$ we have

$$(\mathcal{F}(w_{j,k}))(s) = \frac{e^{-ik2^{-j}s}}{\sqrt{\mu(2^j G)}}\chi_{2^j G}(s),\ \ s \in \check{\mathbb{R}} \sim \mathbb{R}, \tag{2}$$

and $\{\mathcal{F}(w_{j,k})\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$, we get that $\bigcup_{k\in\mathbb{Z}} 2^k G \sim \bigcup_{k\in\mathbb{Z}} 2^k G_1 \sim \mathbb{R}$, which proves that G_1 is a 2-dilation generator of a partition of $\mathbb{R}$.

We assert that for every $j, k \in \mathbb{Z}$, $j \neq k$, the set $(G+2j\pi)\cap(G+2k\pi)$ is a null set. If not, there exists $\ell \in \mathbb{Z}\backslash\{0\}$ such that $\mu(G \cap (G + 2\ell\pi)) > 0$. Set $F = G \cap (G + 2\ell\pi)$, so $F \cup (F - 2\ell\pi) \subset G$. Since $\mu(F) > 0$, there exists a subset F_1 of F of positive measure such that $F_1 \cap (F_1 - 2\ell\pi) = \emptyset$. Thus

$$\|\chi_{F_1} - \chi_{(F_1-2\ell\pi)}\| = \sqrt{2\mu(F_1)} > 0. \tag{3}$$

On the other hand, using (2) and the fact that the family $\{2^jG\}_{j\in\mathbb{Z}}$ is almost disjoint, we have

$$< \mathcal{F}(w_{j,k}), \chi_{F_1} - \chi_{(F_1-2\ell\pi)} >= \int_{\mathbb{R}} \frac{e^{-i2^{-j}ks}}{\sqrt{\mu(2^jG)}} \chi_{2^jG}(s) \left(\chi_{F_1}(s) - \chi_{(F_1-2\ell\pi)}(s)\right) ds =$$

$$\frac{\delta_{j,0}}{\sqrt{\mu(G)}} \int_G e^{-iks}(\chi_{F_1}(s) - \chi_{(F_1-2l\pi)}(s))ds = \frac{\delta_{j,0}}{\sqrt{\mu(G)}} \left(\int_{F_1} e^{-iks}ds - \int_{F_1-2l\pi} e^{-iks}ds \right) = 0,\ j,k\in\mathbb{Z},$$

which contradicts (3). Thus $(G+2j\pi)\cap(G+2k\pi) \sim (G_1+2j\pi)\cap(G_1+2k\pi)$ is a null set for all $j\neq k$ $(j,k\in\mathbb{Z})$. Thus, we can find a subset G' of G_1 such that $G'\sim G_1$ and the family $\{G'+2j\pi\}_{j\in\mathbb{Z}}$ is disjoint. Since $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}(E'+2k\pi)$ (with $E'=[0,2\pi)$), if we define $\varphi: G'\to E'$ by $\varphi(s) = s-2k\pi$ if $s\in G'_k := G'\cap(E'+2k\pi)$, φ becomes an injective measurable map. Then from (2) we obtain that

$$\delta_{k,0}\sqrt{\mu(G)} = \int_{\mathbb{R}} e^{iks}\chi_G(s)ds = \int_{G'} e^{iks}ds = \sum_{k\in\mathbb{Z}} \int_{G'_k} e^{iks}ds =$$

$$\sum_{k\in\mathbb{Z}} \int_{\varphi(G'_k)} e^{iks}ds \int_{\varphi(G')} e^{iks}ds = \sqrt{2\pi}\int_{E'} (1/\sqrt{2\pi})e^{iks}\chi_{\varphi(G')}(s)ds.$$

Since the family of functions $\{s\to(1/\sqrt{2\pi})e^{iks}\chi_{E'}(s)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(E')$, we conclude from the above computation that $\chi_{\varphi(G')}$ is the constant function $(\sqrt{\mu(G)/(2\pi)})\chi_{E'}$ in $L^2(E')$. Hence $\mu(G)=2\pi$ and $\varphi(G')\sim E'$, so

$$\bigcup_{k\in\mathbb{Z}}(G'+2k\pi) = \bigcup_{k\in\mathbb{Z}}\left(\cup_{j\in\mathbb{Z}}(G'_j+2k\pi)\right) = \bigcup_{j\in\mathbb{Z}}\left(\cup_{k\in\mathbb{Z}}(G'_j+2k\pi)\right) =$$

$$\bigcup_{j\in\mathbb{Z}}\left(\cup_{k\in\mathbb{Z}}(\varphi(G'_j)+2(k+j)\pi)\right) = \bigcup_{j\in\mathbb{Z}}\left(\cup_{\ell\in\mathbb{Z}}(\varphi(G'_j)+2\ell\pi)\right) = \tag{4}$$

$$\bigcup_{\ell\in\mathbb{Z}}\left(\cup_{j\in\mathbb{Z}}(\varphi(G'_j)+2\ell\pi)\right) \sim \bigcup_{\ell\in\mathbb{Z}}(E'+2\ell\pi) = \mathbb{R}.$$

This proves that G' is a 2π-translation generator of a partition of $\mathbb{R}$. Since $G'\subset G_1$ and $G'\sim G$, G' is also a 2-dilation generator of a partition of $\mathbb{R}$, so (a) implies (b).

We prove now that (b) implies (c). Thus, suppose that the set G' is as in part (b). We consider the equivalence relation ρ on the null set $\mathbb{R}\backslash(\bigcup_{k\in\mathbb{Z}}2^kG'\cup\{0\})$ defined by $s_1\ \rho\ s_2$ if s_1/s_2 is an integral power of 2. By the axiom of choice there exists a subset $\widetilde{G}$ of $\mathbb{R}\backslash\bigcup_{k\in\mathbb{Z}}2^kG'$ such that $\widetilde{G}$ contains exactly one point from each equivalence class (mod ρ). Clearly $\widetilde{G}$ is a null set and the set $\check{G} = G'\cup\widetilde{G}$ has the property that the family $\{2^k\check{G}\}_{k\in\mathbb{Z}}$ is a veritable measurable partition of $\mathbb{R}\backslash\{0\}$. Hence if we define $\psi: E\to\check{G}$ by $\psi(s) = 2^{-k}s$ for $s\in E\cap(2^k\check{G})$, ψ is a measurable bijection, and so since $\check{G}\sim G$, E is dilation congruent modulo 2π to G. Similarly one shows that E is translation congruent modulo 2π to G.

To establish the implication $(c) \Rightarrow (a)$, let $\psi : E \to \psi(E) \sim G$ be the measurable bijection which satisfies $\psi(s)/s \in \{2^k : k \in \mathbb{Z}\}$ for every $s \in E$. If we consider the measurable sets $E_k := \{s \in E : \psi(s)/s = 2^k\}$, $k \in \mathbb{Z}$, then $2^j E_\ell \cap 2^k E_m \subset 2^j E \cap 2^k E = \emptyset$ for all $\ell, m \in \mathbb{Z}$ and all $j, k \in \mathbb{Z}$, $j \neq k$. Thus for $j \in \mathbb{Z}\backslash\{0\}$ fixed we have

$$2^j \psi(E) \cap \psi(E) = \left(\bigcup_{k \in \mathbb{Z}} 2^{j+k} E_k \right) \cap \left(\bigcup_{k \in \mathbb{Z}} 2^k E_k \right) = \\ \bigcup_{k,\ell \in \mathbb{Z}} 2^{j+k} E_k \cap 2^\ell E_\ell = \bigcup_{\ell \in \mathbb{Z}} 2^\ell (E_{\ell-j} \cap E_\ell) = \emptyset. \tag{5}$$

This implies that the family $\{2^j G\}_{j \in \mathbb{Z}}$ is almost disjoint, and a computation like (4) shows that $\bigcup_{j \in \mathbb{Z}} 2^j G \sim \mathbb{R}$. Therefore by (2) it suffices to show that the family of functions

$$\{s \to (1/\sqrt{\mu(G)}) e^{iks} \chi_G(s)\}_{k \in \mathbb{Z}} \tag{6}$$

is an orthonormal basis for $L^2(G)$. For this purpose let $\varphi : E \to \varphi(E) \sim G$ be a measurable bijection which satisfies $\varphi(s) - s \in \{2k\pi : k \in \mathbb{Z}\}$ for every $s \in E$. We consider $E^k := \{s \in E : \varphi(s) - s = 2k\pi\}$, $k \in \mathbb{Z}$, and an arbitrary function $f \in L^2(G)$. Then

$$\int_G f(s) e^{-iks} ds = \int_{\varphi(E)} f(s) e^{-iks} ds = \\ \sum_{\ell \in \mathbb{Z}} \int_{\varphi(E^\ell)} f(s) e^{-iks} ds = \sum_{\ell \in \mathbb{Z}} \int_{E^\ell + 2\ell\pi} f(s) e^{-iks} ds = \\ \sum_{\ell \in \mathbb{Z}} \int_{E^\ell} f(t + 2\ell\pi) e^{-ikt - 2ik\ell\pi} dt = \int_E f(\varphi(t)) e^{-ikt} dt, \; k \in \mathbb{Z}. \tag{7}$$

First, letting $f(s) = e^{ijs}$ in (7), we get that the family (6) is orthonormal (since $\mu(G) = \mu(\varphi(E)) = \sum_{k \in \mathbb{Z}} \mu(E^k + 2k\pi) = \mu(E) = 2\pi$). Secondly, (7) tells us that the only element of $L^2(G)$ that is orthogonal to the family (6) is $f = 0$. Thus the family (6) is an orthonormal basis for $L^2(G)$, and the lemma is proved. ■

Now let $G \in \mathbb{L}$ be an arbitrary wavelet set. Then we know from Lemma 1.1(b) that there exists a wavelet set $G' \sim G$ such that the families $\{G' + 2k\pi\}_{k \in \mathbb{Z}}$ and $\{2^k G'\}_{k \in \mathbb{Z}}$ are disjoint and satisfy

$$\bigcup_{k \in \mathbb{Z}} (G' + 2k\pi) \sim \mathbb{R}, \qquad \bigcup_{k \in \mathbb{Z}} 2^k G' \sim \mathbb{R}.$$

It turns out to be very convenient for the constructions to follow to know something stronger, namely, that G' may be chosen to also satisfy

$$\bigcup_{k\in\mathbb{Z}}(G'+2k\pi)=\mathbb{R},\qquad \bigcup_{k\in\mathbb{Z}}2^kG'=\mathbb{R}\backslash\{0\}. \tag{8}$$

Strangely enough, this is always possible, as is established by the following result.

Theorem 1.2. *Let G be a wavelet set. Then there exists a (wavelet) set $G'\in\mathbb{L}$ such that $G'\sim G$, G' is a 2-dilation generator of a partition of $\mathbb{R}$ and a 2π-translation generator of another partition of $\mathbb{R}$, and , furthermore, the equalities in (8) hold. Moreover, there exist unique measurable bijections $\hat\varphi$ and $\hat\psi$ of E onto G' satisfying the conditions*

$$\hat\varphi(s)-s\in\{2k\pi:k\in\mathbb{Z}\},\qquad \hat\psi(s)/s\in\{2^k:k\in\mathbb{Z}\},\quad s\in E. \tag{9}$$

Proof. According to part (c) in Lemma 1.1 there exist measurable bijections $\varphi:E\to\varphi(E)\sim G$ and $\psi:E\to\psi(E)\sim G$ such that $\varphi(s)-s\in\{2k\pi:k\in\mathbb{Z}\}$ and $\psi(s)/s\in\{2^k:k\in\mathbb{Z}\}$ for every $s\in E$. Let $G_1=\varphi(E)\cap\psi(E)$ and inductively define the sequences $\{G_n\}_{n\in\mathbb{N}}$ and $\{E_n\}_{n\in\mathbb{N}}$ of sets by

$$E_n=\varphi^{-1}(G_n)\cap\psi^{-1}(G_n),\ n\in\mathbb{N},\qquad G_{n+1}=\varphi(E_n)\cap\psi(E_n),\ n\in\mathbb{N}\backslash\{1\}. \tag{10}$$

It is easy to see that both sequences $\{G_n\}$, $\{E_n\}$ are decreasing and that $E_n\sim E$, $G_n\sim G$ for every n in $\mathbb{N}$. Let $\hat E:=\bigcap_{n\in\mathbb{N}}E_n$, $\hat G:=\bigcap_{n\in\mathbb{N}}G_n$, and observe that

$$\mu(\hat E)=\lim_n\mu(E_n)=2\pi=\lim_n\mu(G_n)=\mu(\hat G), \tag{11}$$

from which we get that $\hat E\sim E$ and $\hat G\sim G$. We assert that $\varphi(\hat E)=\hat G$ and $\psi(\hat E)=\hat G$. We first show that $\varphi(\hat E)\subset\hat G$. Consider $s\in\hat E$ and fix $n\in\mathbb{N}\backslash\{1\}$. Since $s\in E_n$ there exists $t\in G_n$ such that $s=\varphi^{-1}(t)$ and there exists $\xi\in E_{n-1}$ such that $t=\psi(\xi)$. Hence $\varphi(s)=\psi(\xi)\in\psi(E_{n-1})$, and since $\varphi(s)\in\varphi(E_n)\subset\varphi(E_{n-1})$, it follows from (10) that $\varphi(s)\in G_n$, and thus that $\varphi(s)\in\hat G$ and $\psi(\hat E)\subset\hat G$. Analogously one establishes that $\psi(\hat E)\subset\hat G$, $\varphi^{-1}(\hat G)\subset\hat E$ and $\psi^{-1}(\hat G)\subset\hat E$, which shows that $\varphi(\hat E)=\hat G=\psi(\hat E)$.

We define now $G'=\hat G\cup(E\backslash\hat E)$. From (11) we get $G'\sim G$, and if we define $\hat\varphi$ [resp. $\hat\psi$] $:E\to G'$ to be φ [resp. ψ] on $\hat E$ and the identity function on $E\backslash\hat E$, we obtain that $\hat\varphi$ and $\hat\psi$ are measurable bijections (with range G') satisfying (9). (To verify the injectivity of $\hat\varphi$ and $\hat\psi$, let $s_1\in E\backslash\hat E$, $s_2\in\hat E$, and $\hat\varphi(s_1)=\hat\varphi(s_2)$ [resp. $\hat\psi(s_1)=\hat\psi(s_2)$]. Then $\varphi(s_2)$[resp. $\psi(s_2)$] $=s_1\in E$. Since $s_2,\psi(s_2)\in E$ and $s_2-\psi(s_2)$ is an integral multiple of 2π, we must have $\varphi(s_2)=s_2=s_1$ and similarly for ψ.) Two computations similar to (4) show that both equalities in (8) hold, and two computations similar to (5) show that the sequences $\{G'+2k\pi\}_{k\in\mathbb{Z}}$ and $\{2^kG'\}_{k\in\mathbb{Z}}$ are disjoint. Finally, the uniqueness of $\hat{\varphi}$ and $\hat{\psi}$ follows immediately from (9) and the fact that G' is both a 2-dilation generator and a 2π-translation generator of partitions of $\mathbb{R}$. ■

Definition 1.3. *If $G \subset \mathbb{R}$ is any wavelet set and G' is related to G as in Theorem 1.2, we say that G' is a **regularization** of G and that G' is a **regularized** wavelet set. (Easy examples show that such a G' need not be unique.)*

2. The main result.

The following concept will be very useful. If $(X, \mathbb{M})$ is any measurable space and $g : X \to X$ is a measurable bijection with measurable inverse, a measurable set $W \subset X$ is called a *wandering set* for g if the family $\{g^{(k)}(W)\}_{k\in\mathbb{Z}}$ is disjoint and $X = \bigcup_{k\in\mathbb{Z}} g^{(k)}(W)$, where $g^{(k)}$ is the identity function on X when $k = 0$ and the composition of g [resp. g^{-1}] with itself $|k|$ times when $k > 0$ [resp. $k < 0$]. (In particular, if $X = \emptyset$, then $W = \emptyset$ is a wandering set for g.)

With these preliminaries disposed of, we now turn to our principal construction. Let F be any wavelet set, let F' be a regularization of F, and let $\hat{\varphi} : E \to F'$, $\hat{\psi} : E \to F'$ be the measurable bijections given by Theorem 1.2. Since $\hat{\varphi}^{-1}$ and $\hat{\psi}^{-1}$ are measurable too (via (9)), the function $h : E \to E$ defined by $h = \hat{\varphi}^{-1} \circ \hat{\psi}$ and its inverse are measurable bijections. We consider a measurable partition $E = E_1 \cup E_2 \cup E_3 \cup ... \cup E_\infty$ of E, where

$$E_n = \{t \in E : h^{(n)}(t) = t, h^{(k)}(t) \neq t, 1 \leq k < n\},\ n \in \mathbb{N}, \qquad E_\infty = E \setminus \left(\bigcup_{k=1}^{\infty} E_k\right). \tag{12}$$

For each $n \in \mathbb{N} \cup \{\infty\}$, h maps each E_n onto itself and $h_{|E_n}$ is a measurable bijection with measurable inverse. The following is our principal theorem.

Theorem 2.1. *Let E be the Littlewood-Paley wavelet set as above, F any wavelet set, and F' a regularization of F. Suppose there exists a wandering set W for the function $h_{|E_\infty}$ defined above. Then there exists a family of (regularized) wavelet sets $\{G_t\}_{t\in[0,1]}$ such that $G_0 = E$, $G_1 \bigtriangledown F'$ is at most countable, and $G_t \subset E \cup F'$ for all $t \in [0,1]$. Moreover, the function $t \to \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}}\chi_{G_t})$ defined on $[0,1]$ is continuous (with respect to the norm topology on $L^2(\mathbb{R})$) and consists entirely of wavelets.*

Proof. Let $\hat{\varphi}$ and $\hat{\psi}$ as above. For $t \in [0,1]$ and $n \in \mathbb{N}$ we define the sets

$$P^t = [-2\pi, -(2-t)\pi) \cup [(2-t)\pi, 2\pi), \quad W_t = W \cap P^t, \quad P_n^t = E_n \cap P^t,$$
$$\Omega_\infty^t = \bigcup_{k\in\mathbb{Z}} h^{(k)}(W_t), \quad \Omega_n^t = \bigcup_{k=0}^{n-1} h^{(k)}(P_n^t), \quad \Omega_t = \Omega_\infty^t \cup \bigcup_{k=2}^{\infty} \Omega_k^t, \tag{13}$$

where the E_n are as in (12). Then the desired family of wavelet sets is given by

$$G_t := (E \backslash \Omega_t) \cup \hat{\varphi}(\Omega_t), \quad t \in [0,1]. \tag{14}$$

Clearly the sets W_t, P_n^t, Ω_∞^t, Ω_n^t, Ω_t (and consequently G_t) are measurable for all $t \in [0,1]$ and $n \in \mathbb{N}$. It is easy to see that $G_0 = E$ (since $P^0 = \emptyset$). To show that $G_1 \bigtriangledown F'$ is at most countable, we note that $\hat{\varphi}(s) = s$ for all s in E_1 except possibly a countable set. (Indeed, if $s \in E_1$, then $h(s) = s$ and this leads to the equation $\hat{\varphi}(s) = s + 2l\pi = 2^m s = \hat{\psi}(s)$ for some $l, m \in \mathbb{Z}$. If $m = 0$ then $l = 0$, which shows that $\hat{\varphi}(s) = s$. If $m \neq 0$ then $s = -2l\pi/(2^m - 1)$, and the set of such points is countable.) A calculation using (13) shows that $G_1 = (E \backslash \Omega_1) \cup \hat{\varphi}(\Omega_1) = E_1 \cup \hat{\varphi}(E \backslash E_1) = E_1 \cup (F' \backslash \hat{\varphi}(E_1))$, so $G_1 \bigtriangledown F'$ is countable. From (14) we see that $G_t \subset E \cup F'$ for all $t \in [0,1]$.

To check that each G_t is a wavelet set, by Lemma 1.1 it suffices to show that E is translation congruent modulo 2π and dilation congruent modulo 2 to G_t. For this purpose let us observe that $h(\Omega_t) = \Omega_t$ and thus $\hat{\varphi}(\Omega_t) = \hat{\psi}(\Omega_t)$ for all $t \in [0,1]$. Moreover we have $\hat{\varphi}(\Omega_t) \cap E = \emptyset$ for all $t \in [0,1]$. (Indeed, if $\hat{\varphi}(s) \in E$ for some s in Ω_t, then $\hat{\varphi}(s) = s$ by (9) and the fact that $\{E + 2k\pi\}_{k \in \mathbb{Z}}$ is a disjoint family. Hence $s \in E \cap F'$, and since $\hat{\psi}(s) \in F'$, we must have $\hat{\psi}(s) = s$ (since $\{2^k F'\}_{k \in \mathbb{Z}}$ is a disjoint family). This implies that $h(s) = (\hat{\varphi}^{-1} \circ \hat{\psi})(s) = s$ and hence $s \in E_1$. But this contradicts the fact that $\Omega_t \cap E_1 = \emptyset$ by the definition of Ω_t.) We consider the measurable maps φ_t, $\psi_t : E \to G_t$ defined to be the identity function on $E \backslash \Omega_t$ and by $\varphi_t = \hat{\varphi}$, $\psi_t = \hat{\psi}$ on Ω_t for all $t \in [0,1]$. Since $G_t = (E \backslash \Omega_t) \cup \hat{\varphi}(\Omega_t) = (E \backslash \Omega_t) \cup \hat{\psi}(\Omega_t)$ for all $t \in [0,1]$, and $(E \backslash \Omega_t) \cap \hat{\varphi}(\Omega_t) = (E \backslash \Omega_t) \cap \hat{\psi}(\Omega_t) = \emptyset$, φ_t and ψ_t are injective and map E onto G_t. It is clear that for all $t \in [0,1]$ and all $s \in E$, $\varphi_t(s) - s$ is an integral multiple of 2π and $\psi_t(s)/s$ is an integral power of 2. As in the proof of Theorem 1.2, one concludes easily that all the sets G_t are regularized wavelets sets.

Since $\mathcal{F}$ is a unitary operator, to check that the function $t \to \mathcal{F}^{-1}(\frac{1}{\sqrt{2\pi}} \chi_{G_t})$ is continuous, it suffices to show that the function $t \to \chi_{G_t}$ is continuous on $[0,1]$. For this purpose let t_1, $t_2 \in [0,1]$, and, without loss of generality, suppose that $t_1 < t_2$. Then $P^{t_1} \subset P^{t_2}$ and consequently $\Omega_{t_1} \subset \Omega_{t_2}$. Thus

$$\|\chi_{G_{t_1}} - \chi_{G_{t_2}}\|_{L^2}^2 = \|\chi_{G_{t_1} \bigtriangledown G_{t_2}}\|_{L^2}^2 = \mu(\Omega_{t_2} \backslash \Omega_{t_1}) + \mu(\varphi(\Omega_{t_2} \backslash \Omega_{t_1})). \tag{15}$$

Since $\hat{\varphi}$ is μ-measure preserving, (15) becomes

$$\|\chi_{G_{t_1}} - \chi_{G_{t_2}}\|_{L^2}^2 = 2\left(\mu(\Omega_{t_2}) - \mu(\Omega_{t_1})\right). \tag{16}$$

The following lemma completes the proof of Theorem 2.1.

Lemma 2.2. *The function ω defined by $\omega(t) = \mu(\Omega_t)$, $t \in [0,1]$, is continuous on $[0,1]$.*

Proof. We begin by defining the functions ω_n on $[0,1]$ by

$$\omega_0(t) = \mu(\Omega^t_\infty), \qquad \omega_n(t) = \mu(\Omega^t_n), \quad n \in \mathbb{N}.$$

From (13) we see that

$$\omega(t) = \omega_0(t) + \sum_{k=2}^{\infty} \omega_k(t), \quad t \in [0,1]. \tag{17}$$

Since $h(E_n) = E_n$ for all $n \in \mathbb{N}$, we obtain

$$\sum_{k=2}^{\infty} \omega_k(t) \leq \sum_{k=2}^{\infty} \mu(E_k) \leq \mu(E) = 2\pi, \quad t \in [0,1],$$

so the series in (17) converges uniformly on $[0,1]$. Thus it suffices to show that for $n = 0,\ 1, \ldots$, the function ω_n is continuous on $[0,1]$. For $n = 1,\ 2, \ldots$, we note that if $0 \leq t_1 < t_2 \leq 1$, then $\Omega^{t_2}_n \setminus \Omega^{t_1}_n \subset \bigcup_{k=0}^{n-1} h^{(k)}(P^{t_2}_n \setminus P^{t_1}_n)$, and consequently we have

$$0 \leq \omega_n(t_2) - \omega_n(t_1) \leq \sum_{k=0}^{n-1} \mu(h^{(k)}(P^{t_2}_n \setminus P^{t_1}_n)), \quad n \in \mathbb{N}.$$

Since $\mu(P^{t_2}_n \setminus P^{t_1}_n) \leq 2\pi|t_1 - t_2|$ (cf. (13)), to establish the continuity of the ω_n, $n \in \mathbb{N}$, it is sufficient to show that for $k \in \mathbb{Z}$ the measure ν_k defined on $(E, \mathbb{L} \cap E)$ by $\nu_k(L) = \mu(h^{(k)}(L))$, $L \in \mathbb{L} \cap E$, $k \in \mathbb{Z}$, is absolutely continuous with respect to $\mu_{|(\mathbb{L}\cap E)}$. Moreover, using (13) and the facts that $W_t \subset W$ and W is a wandering set for $h_{|E_\infty}$, we get

$$\omega_0(t) = \sum_{k \in \mathbb{Z}} \mu(h^{(k)}(W_t)) \tag{18}$$

and

$$\sum_{k \in \mathbb{Z}} \mu(h^{(k)}(W_t)) \leq \sum_{k \in \mathbb{Z}} \mu(h^{(k)}(W)) = \mu(E_\infty) \leq 2\pi, \quad t \in [0,1].$$

This shows that the convergence of the series in (18) is uniform on $[0,1]$, and since

$$\mu(W_{t_2} \setminus W_{t_1}) \leq 2\pi|t_1 - t_2|, \quad t_1, t_2 \in [0,1],$$

the continuity of ω_0 will also follow from the absolute continuity $[\mu]$ of the ν_k, $k \in \mathbb{Z}$.

In order to check the relations $\nu_k << \mu_{|\mathbb{L}\cap E}$, we observe that the measure ν_1 is mutually absolutely continuous with respect to (i.e., equivalent to) $\mu_{|\mathbb{L}\cap E}$. Indeed, let L be a measurable subset of E such that $\mu(L) = 0$. It is clear from (9) that for every s in E we get $h(s) = 2^m s + 2k\pi$ for some integers m and k. Thus we consider the measurable partition $E = \bigcup_{i,j \in \mathbb{Z}} H^{ij}$ of E, where $H^{ij} = \{s \in E : h(s) = 2^i s + 2j\pi\}$. Since h is a measurable bijective map, $h(L)$ is the disjoint union

$$\bigcup_{i,j \in \mathbb{Z}} h(L \cap H^{ij}) = \bigcup_{i,j \in \mathbb{Z}} \left(2^i (L \cap H^{ij}) + 2j\pi\right).$$

This implies that

$$\nu_1(L) = \sum_{i,j\in\mathbb{Z}} \mu\left(2^i(L\cap H^{ij}) + 2j\pi\right) = \sum_{i,j\in\mathbb{Z}} 2^i\mu(L\cap H^{ij}) = 0. \tag{19}$$

Reciprocally, if L is a measurable subset of E such that $\nu_1(L) = 0$, we obtain from (19) that $\mu(L\cap H^{ij}) = 0$ for every $i,j \in \mathbb{Z}$. Hence $\mu(L) = \sum_{i,j\in\mathbb{Z}} \mu(L\cap H^{ij}) = 0$. Thus ν_1 and μ vanish simultaneously on $\mathbb{L}\cap E$, and by virtue of the relationships between ν_1, ν_k and μ, the same is true for all ν_k, $k\in\mathbb{Z}$. Since the ν_k and μ are all finite measures on $(E, \mathbb{L}\cap E)$, for each fixed $k\in\mathbb{Z}$ and for each $\varepsilon > 0$, there exists $\delta_k > 0$ such that $\nu_k(L) \le \varepsilon$ whenever $\mu(L) \le \delta_k$ (see, e.g., [1], p. 169). This is what was needed, so the proof is complete. ■■

Remark 2.3. In Theorem 2.1, E is the Littlewood-Paley wavelet set, but the content of the theorem is not altered if we replace E by any regularized wavelet set.

Corollary 2.4. *Let F be a wavelet set for which the set E_∞ defined in the preamble to Theorem 2.1 is a null set (equivalently, for which almost every point of E has a finite orbit under the corresponding function h). Then there exists a wandering set W for $h_{|E_\infty}$, and thus the conclusions of Theorem* 2.1 *hold.*

Proof. If E_∞ is void, then $\emptyset$ is a wandering set for $h_{|E_\infty}$. If $E_\infty \neq \emptyset$, we consider the equivalence relation δ on E_∞ given by $s\ \delta\ s'$ if $h^{(k)}(s) = s'$ for some integer k. By the Axiom of Choice there exists a subset W of E_∞ such that W contains exactly one point from each equivalence class (mod δ). By hypothesis it follows that W is a measurable null set. Clearly, by construction, $h^{(k)}(W)\cap W = \emptyset$ for all $k\in\mathbb{N}$ and $E_\infty = \bigcup_{k\in\mathbb{Z}} h^{(k)}(W)$. Hence W is a wandering set for $h_{|E_\infty}$, and this proves the corollary. ■

There are several examples of wavelets sets to which we can apply Corollary 2.4.

Example 2.5. First we consider the class of all wavelet sets F such that (E,F) is an interpolation pair (see [2], p. 56). (Indeed, in the notation of [2], if $(\sigma_E^F)^2$ is the identity function (a.e.) on $\mathbb{R}$ one may observe that $(\sigma_E^F)_{|F'} = \hat{\varphi}\circ\hat{\psi}^{-1}$ (a.e.) in our notation, and thus $h^{(2)}$ is the identity function (a.e.) on E. This implies that $E \sim E_1\cup E_2$ and hence that E_∞ is a null set.) Consequently we obtain the following result first mentioned in [2]: any two wavelets associated with an interpolation pair of wavelet sets can be connected by an arc of wavelets.

Example 2.6. There are cases (see [2], p. 83) when $(\sigma_E^F)^3$ is the identity function on $\mathbb{R}$. Clearly, these cases are also covered by Corollary 2.4. Indeed, Corollary 2.4 applies whenever there exists an $n \in \mathbb{N}$ such that $h^{(n)}$ is the identity function (a.e.) on E, since in this case $E \sim \bigcup_{j=1}^{n} E_j$. In fact, we can prove somewhat more.

Corollary 2.7. *Let F be a wavelet set such that the function h defined in the preamble to Theorem 2.1 has the property that $h^{(k)}$ is continuous on E for some $k \in \mathbb{Z}\backslash\{0\}$. Then there exists a wandering set W for $h_{|E_\infty}$, and thus the conclusions of Theorem 2.1 hold.*

Proof. We write g for $h^{(k)}$, and recall that g and g^{-1} are measurable and injective maps of E onto itself. Since g maps connected sets to connected sets, we must have $g([\pi, 2\pi)) = [-2\pi, -\pi)$ or $g([\pi, 2\pi)) = [\pi, 2\pi)$. In any case, by squaring if necessary, we may supppose that g maps $[\pi, 2\pi)$ onto itself and thus that $g(\pi) = \pi$ and g is strictly increasing on $[\pi, 2\pi)$. Hence $\lim_{s \to 2\pi} g(s) = 2\pi$. In other words, g can be extended to be a continuous bijection on $[\pi, 2\pi]$, and thus a homeomorphism. Consequently g is a homeomorphism of $[\pi, 2\pi)$ onto itself, and similarly for g on $[-2\pi, -\pi)$. Let C^+ be the set of fixed points of $g_{|[\pi,2\pi)}$ (which includes π), and let $U^+ = [\pi, 2\pi) \setminus C^+$. If $U^+ = \emptyset$, then $\emptyset$ is a wandering set for $g_{|U^+}$. Otherwise U^+ is a nonempty open subset of $\mathbb{R}$ which can be written uniquely as a countable union $U^+ = \bigcup_{j \in J}(a_j, b_j)$ of disjoint open intervals. Clearly, $a_j,\ b_j \in C^+ \cup \{2\pi\}$ and $g((a_j, b_j)) = (a_j, b_j)$ for all j. Consequently, U^+ consists exactly of those points $s \in [\pi, 2\pi)$ such that the orbit of s under g (and therefore under h) is infinite. We now exhibit a wandering set W^+ for $g_{|U^+}$. For each $j \in J$ we choose a point (say, the midpoint) $s_j \in (a_j, b_j)$ and define $W^+ = \bigcup_j I_j$, where $I_j = [s_j, g(s_j))$ if $s_j < g(s_j)$ and $I_j = [g(s_j), s_j)$ if $g(s_j) < s_j$. It is easy to see that the family of sets $\{g^{(k)}(W^+)\}_{k \in \mathbb{Z}}$ is disjoint and $\bigcup_{k \in \mathbb{Z}} g^{(k)}(W^+) = U^+$. In a similar way we construct a wandering set W^- for $g_{|U^-}$ where $U^- \subset [-2\pi, -\pi)$ is defined analogously to U^+ , and we write $\widetilde{W} = W^+ \cup W^-$. Clearly $\widetilde{W}$ is a wandering set for $g_{|U^+ \cup U^-}$ (and $(g^{-1})_{|U^+ \cup U^-}$). The following lemma applied to $X = U^+ \cup U^-$, $\mathbb{M} = \mathbb{L} \cap X$, and $\nu = \mu_{|X}$ completes the proof of the corollary.

Lemma 2.8. *Let $(X, \mathbb{M}, \nu)$ be a complete, finite measure space, and let $h : X \to X$ be a measurable bijection with measurable inverse which preserves sets of measure zero and has the property that for some $n \in \mathbb{N}$, $h^{(n)}$ has a wandering set. Then h also has a wandering set.*

Proof. Suppose W is a wandering set for $h^{(k_0)}$ $(k_0 > 1)$, and consider the collection $\mathcal{S}$ of subsets of W defined by

$$\mathcal{S} := \{S \subset W : S \in \mathbb{M} \text{ and } \{h^{(j)}(S)\}_{j \in \mathbb{Z}} \text{ is a disjoint family}\}.$$

Obviously $\emptyset \in \mathcal{S}$, and we consider the partial ordering defined on $\mathcal{S}$ by $S_1 \prec S_2$ if $S_1 \subset S_2$ and $\nu(S_2 \backslash S_1) > 0$. Let $\mathcal{C} = \{S_\alpha\}_{\alpha \in A}$ be a chain in $\mathcal{S}$. Then A is at most countable since W cannot contain uncountably many disjoint sets of positive measure, and an easy calculation shows that $U = \bigcup_{\alpha \in A} S_\alpha \in \mathcal{S}$. Thus U is an upper bound for the chain $\mathcal{C}$, and by Zorn's Lemma, $\mathcal{S}$ contains a maximal element M. By definition, the family $\{h^{(j)}(M)\}_{j\in\mathbb{Z}}$ is disjoint, and we assert that $\nu\left(X \setminus \bigcup_{j\in\mathbb{Z}} h^{(j)}(M)\right) = 0$. Suppose, for the moment, that this has been established. Then, if $X \setminus \bigcup_{j\in\mathbb{Z}} h^{(j)}(M) \neq \emptyset$, we put an equivalence relation on this set of measure zero as in the proof of Corollary 2.4, and using the Axiom of Choice as in that corollary and the completeness of $(X, \mathbb{M}, \nu)$, we obtain a set T of measure zero such that $M \cup T$ is a wandering set for h.

Thus it suffices to verify that $X \setminus \bigcup_{j\in\mathbb{Z}} h^{(j)}(M)$ has measure zero. Suppose not. Then $R = W \setminus (\bigcup_{j\in\mathbb{Z}} h^{(j)}(M))$ must also have positive measure. (Otherwise, by hypothesis, $X = \bigcup_{\ell\in\mathbb{Z}} h^{(\ell k_0)}(W)$ is almost contained in $\bigcup_{j\in\mathbb{Z}} h^{(j)}(M)$.) Using the fact that

$$\bigcup_{\ell\in\mathbb{Z}} h^{(k_0\ell+j)}(M) = X, \quad j = 1, ..., k_0 - 1,$$

we may choose integers $n_1, ..., n_{k_0-1}$ by induction so that the (measurable) set

$$Q = R \cap h^{(k_0 n_1+1)}(W) \cap h^{(k_0 n_2+2)}(W) \cap ... \cap h^{(k_0 n_{k_0-1}+k_0-1)}(W)$$

has positive measure, and since $Q \subset R$ and $R \cap (\bigcup_{j\in\mathbb{Z}} h^{(j)}(M)) = \emptyset$, we have

$$h^{(m)}(Q) \cap (\bigcup_{j\in\mathbb{Z}} h^{(j)}(M)) = \emptyset, \quad m \in \mathbb{Z}.$$

Thus $M \cup Q \in \mathcal{S}$, which contradicts the fact that M is a maximal element of $\mathcal{S}$, so the proof is complete. ■■

Remark 2.8. One can prove that, in fact, the hypothesis in Lemma 2.8 can be weakened to the assumption that $(X, \mathbb{M})$ is simply a measurable space. The proof is very computational (so we don't include it here), and is based upon checking that the set

$$\hat{W} = W \cap \left(\bigcup_{(j_1,...,j_{k_0-1})\in\mathcal{A}} \bigcap_{\ell=1}^{k_0-1} h^{(j_\ell)}(W) \right),$$

where $\mathcal{A}$ is a subset of the product $(k_0\mathbb{Z} + 1) \times ... \times (k_0\mathbb{Z} + k_0 - 1)$ with certain properties, is a wandering set for h.

Example 2.9. An interesting wavelet set to which Corollary 2.7 applies is Journe's (regularized) wavelet set (cf. [2], p. 45):

$$J = [-\frac{32\pi}{7}, -4\pi) \cup [-\pi, -\frac{4\pi}{7}) \cup [\frac{4\pi}{7}, \pi) \cup [4\pi, \frac{32\pi}{7}).$$

One can easily compute the corresponding function h (constructed from E and J) and obtain that $h(s) = s/4 + 3\pi/2$ for $s \in [-2\pi, -10\pi/7)$, $h(s) = 2s + 4\pi$ if $s \in [-10\pi/7)$, $h(s) = 2s - 4\pi$ for $s \in [\pi, 10\pi/7)$, and $h(s) = s/4 - 3\pi/2$ if $s \in [10\pi/7, 2\pi)$. Obviously h is continuous, but calculations show that E_∞ is not a null set, so Corollary 2.4 doesn't apply. But the set

$$[-9\pi/7, -8\pi/7) \cup [8\pi/7, 9\pi/7)$$

is easily seen to be a wandering set for $h_{|E_\infty}$, and thus Theorem 2.1 applies.

Example 2.10. The following (regularized) wavelet set G (due to Gu [5]), is an *interval* wavelet set (i.e., a wavelet set which is a union of intervals), but even so, for every $k \in \mathbb{Z}$ the function $h^{(k)}$ is not continuous:

$$G := [-640\pi/21, -448\pi/15) \cup [-56\pi/15, -8\pi/3) \cup [-2\pi/3, -10\pi/21) \cup [2\pi/15, 4\pi/15).$$

Indeed, an analysis of the functions $\hat{\varphi}$ and $\hat{\psi}$ allows one to check that h is given by

$$h(s) = \begin{cases} s/4 + 2\pi, & s \in [-2\pi, -40\pi/21), \\ 16s + 32\pi, & s \in [-40\pi/21, -15\pi/8), \\ 16s + 28\pi, & s \in [-15\pi/8, -28\pi/15), \\ 2s + 2\pi, & s \in [-28\pi/15, -3\pi/2), \\ 2s + 4\pi, & s \in [-3\pi/2, -4\pi/3), \\ s/2 + 2\pi, & s \in [-4\pi/3, -\pi), \\ s/4 - 2\pi, & s \in [\pi, 16\pi/15), \\ s/8 - 2\pi, & s \in [16\pi/15, 2\pi). \end{cases} \tag{20}$$

One can trace the orbit $\{h^{(n)}(-2\pi)\}_{n\in\mathbb{N}}$ and observe that it is an infinite set. But from the proof of Corollary 2.7, one knows that if $h^{(k)}$ were continuous for some $k \in \mathbb{Z}$, then -2π would be a fixed point for $h^{(2k)}$. Still, it turns out that $h_{|E_\infty}$ has a wandering set, so Theorem 2.1 applies again. In fact, one can check that $E = E_3 \cup E_4 \cup E_\infty$, where

$$E_3 = [-7\pi/4, -26\pi/15) \cup [-3\pi/2, -22\pi/15) \cup [\pi, 16\pi/15),$$

$$E_4 = [-28\pi/15, -11\pi/6) \cup [-26\pi/15, -5\pi/3) \cup [-22\pi/15, -4\pi/3) \cup [16\pi/15, 4\pi/3),$$

$$E_\infty = [-2\pi, -28\pi/15) \cup [-11\pi/6, -7\pi/4) \cup [-5\pi/3, -3\pi/2) \cup [-4\pi/3, -\pi) \cup [4\pi/3, 2\pi),$$

and the interval $[3\pi/2, 2\pi)$ is a wandering set for $h_{|E_\infty}$.

We have reason to believe that the following conjecture, which we are presently unable to prove, is true.

Conjecture 2.11. Given an arbitrary wavelet set F, there exists a wandering set for the corresponding function $h_{|E_\infty}$ associated with E and F.

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