

# ON SUBWAVELET SETS

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**ABSTRACT:** In this note we give a characterization of subwavelet sets and show that any point  $x \in \mathbb{R} \setminus \{0\}$  has a neighborhood which is contained in a regularized wavelet set. <sup>1</sup>

In [1] the notion of a wavelet set was introduced and in [8] subwavelet sets were considered. Wavelet sets were also introduced independently and simultaneously as the support sets of MSF (Minimally Supported Frequency) wavelets in the sequence of papers [3], [5], and [6]. (See also the recent excellent book [4].) The purpose of this note is to provide a characterization of the subwavelet sets and to use this characterization to prove that every point  $x \in \mathbb{R} \setminus \{0\}$  has a neighborhood contained in a regularized wavelet set. (Regularized wavelet sets are wavelet sets with certain nice properties; see [7].) In particular, this shows that the union of the interiors of all wavelet sets is  $\mathbb{R} \setminus \{0\}$ .

We begin by introducing some preliminary terminology and notation. The measure space under consideration will always be  $\mathbb{R}$  together with its  $\sigma$ -ring  $\mathcal{L}$  of Lebesgue measurable subsets and Lebesgue measure  $\mu$ . Recall (cf. [1]) that a function  $w \in L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathcal{L}, \mu)$  is a wavelet if the family of (equivalence classes of) functions  $\{w_{j,k}\}_{j,k \in \mathbb{Z}}$  defined by

$$w_{j,k}(s) = 2^{j/2} w(2^j s + k), \quad s \in \mathbb{R}, \quad j, k \in \mathbb{Z},$$

is an orthonormal basis for  $L^2(\mathbb{R})$ . A subset  $G$  of  $\mathbb{R}$  with positive measure is a *wavelet set* if  $\frac{1}{\sqrt{\mu(G)}} \chi_G = \mathcal{F}(w)$ , where  $w$  is a wavelet in  $L^2(\mathbb{R})$  and  $\mathcal{F}$  is the Fourier-Plancherel transform on  $L^2(\mathbb{R})$ . A measurable subset  $G$  of  $\mathbb{R}$  is called a *regularized wavelet set* if the family  $\{G + 2k\pi\}_{k \in \mathbb{Z}}$  is a partition of  $\mathbb{R}$  and the family  $\{2^k G\}_{k \in \mathbb{Z}}$  is a partition of  $\mathbb{R} \setminus \{0\}$ . For two measurable subsets  $F$  and  $G$  of  $\mathbb{R}$ , we write  $F \sim G$  if  $\mu(F \nabla G) = 0$ . It is proved in [7] that if  $W$  is any wavelet set, then there exists a regularized wavelet set  $W'$  such that  $W' \sim W$ . A measurable subset  $G$  of  $\mathbb{R}$  is *translation congruent modulo  $2\pi$*  to a (measurable) set  $H \subset \mathbb{R}$  if there exists a measurable bijection  $\varphi : G \rightarrow \varphi(G)$  such that  $\varphi(s) - s$  is an

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integral multiple of  $2\pi$  for every  $s$  in  $G$  and  $\varphi(G) \sim H$ . Analogously,  $G \subset \mathbb{R} \setminus \{0\}$  is said to be *dilation congruent modulo 2* to a (measurable) set  $H$  if there exists a measurable bijection  $\psi : G \rightarrow \psi(G)$  such that  $\psi(s)/s$  is an integral power of 2 for every  $s$  in  $G$  and  $\psi(G) \sim H$ . Let  $\tau : \mathbb{R} \rightarrow E := [-2\pi, -\pi) \cup [\pi, 2\pi)$  be the function defined by  $\tau(x) = x + 2j\pi$ , where  $j$  is the unique integer satisfying  $x + 2j\pi \in E$ , and let  $\delta : \mathbb{R} \setminus \{0\} \rightarrow E$  be the function defined by  $\delta(x) = 2^k x$ , where  $k$  is the unique integer for which  $2^k x \in E$ . For a function  $f : X \rightarrow X$  and  $k \in \mathbb{Z}$  we write  $f^{(0)}$  for the map  $x \rightarrow x$  on  $X$  and  $f^{(k)}$  for the composition of  $f$  [resp.  $f^{-1}$ ] with itself  $|k|$  times if  $k > 0$  [resp.  $k < 0$ ].

**Remark 1.** *In what follows we use the elementary facts that if  $G \in \mathcal{L} \cap E$ , then  $\tau^{-1}(G)$ ,  $\delta^{-1}(G) \in \mathcal{L}$ , and if  $H \in \mathcal{L}$  [resp.,  $H \in (\mathbb{R} \setminus \{0\}) \cap \mathcal{L}$ ], then  $\tau(G) \in \mathcal{L}$  [ $\delta(G) \in \mathcal{L}$ ].*

A measurable subset  $G$  of  $\mathbb{R}$  is called a *subwavelet set* if it is a subset of some regularized wavelet set. Our principal result characterizes measurable subsets of  $\mathbb{R}$  that are subwavelet sets.

**Theorem 2.** *A set  $G \in \mathcal{L}$  is a subwavelet set if and only if there exist sets  $G_1$  and  $G_2$  in  $\mathcal{L}$ , each containing  $G$ , such that*

- (a)  $\tau|_{G_1}$  is measurable bijection of  $G_1$  onto  $E$ ,
- (b)  $\tau|_{G_2}$  is measurable injection of  $G_2$  into  $E$ ,
- (c)  $\delta|_{G_2}$  is measurable bijection of  $G_2$  onto  $E$ , and
- (d)  $\delta|_{G_1}$  is measurable injection of  $G_1$  into  $E$ .

*Proof.* Suppose first that  $G$  is a subset of a regularized wavelet set  $W$ . Define  $G_1 = G_2 = W$ , and observe that (a) – (d) follow from the definition of a regularized wavelet set and Remark 1.

For the sufficiency, suppose that there exist measurable sets  $G_1$  and  $G_2$  containing  $G$  such that (a) – (d) hold. We consider the maps  $h_1, h_2 : E \rightarrow E$  defined by  $h_1 := \delta|_{G_1} \circ (\tau|_{G_1})^{-1}$  and  $h_2 := \tau|_{G_2} \circ (\delta|_{G_2})^{-1}$ . It is clear that  $h_1$  and  $h_2$  are measurable injections. We now construct a new map  $h$  from  $h_1$  and  $h_2$  following the idea of the proof of the Cantor-Bernstein theorem in set theory. To increase the clarity of the presentation we write  $\tilde{E} := E$  and consider  $h_1 : E \rightarrow \tilde{E}$  and  $h_2 : \tilde{E} \rightarrow E$ . We denote  $f = h_2 \circ h_1 : E \rightarrow E$  and  $g := h_1 \circ h_2 : \tilde{E} \rightarrow \tilde{E}$ , and note that these maps are measurable injections by Remark 1. One can check that  $E$  and  $\tilde{E}$  can be partitioned as follows:

$$E = E_0 \dot{\cup} \left( \dot{\bigcup}_{k \in \mathbb{N}} E_k \dot{\cup} E'_k \right),$$

$$\tilde{E} = \tilde{E}_0 \dot{\cup} \left( \dot{\bigcup}_{k \in \mathbb{N}} \tilde{E}_k \dot{\cup} \tilde{E}'_k \right),$$

where

$$\begin{aligned} E_0 &= \bigcap_{j \in \mathbb{N}} f^{(j)}(E), & \tilde{E}_0 &= \bigcap_{j \in \mathbb{N}} g^{(j)}(\tilde{E}), \\ E_k &= f^{(k-1)}(E) \setminus (f^{(k-1)} \circ h_2)(\tilde{E}), & E'_k &= (f^{(k-1)} \circ h_2)(\tilde{E}) \setminus f^{(k)}(E), \quad k \in \mathbb{N}, \end{aligned}$$

and

$$\tilde{E}_k = g^{(k-1)}(\tilde{E}) \setminus (g^{(k-1)} \circ h_1)(E), \quad \tilde{E}'_k = (g^{(k-1)} \circ h_1)(E) \setminus g^{(k)}(\tilde{E}), \quad k \in \mathbb{N}.$$

We define the map  $h : E \rightarrow \tilde{E}$  to be  $h_1$  on  $\hat{E} = E_0 \dot{\cup} \left( \dot{\bigcup}_{k \in \mathbb{N}} E_k \right)$ ,  $h_2^{-1}$  on  $\hat{E}' = \dot{\bigcup}_{k \in \mathbb{N}} E'_k$ . Since  $h_1(E_0) = \tilde{E}_0$  and for  $k \in \mathbb{N}$ ,  $h_1(E_k) = \tilde{E}'_k$ , and  $h_2^{-1}(E'_k) = \tilde{E}_k$ ,  $h$  is a bijection. We define

$$W = (\tau_{|G_1})^{-1}(\hat{E}) \cup (\tau_{|G_2})^{-1}(\hat{E}'). \quad (1)$$

Since  $\hat{E}'$  is a set in the range of  $h_2$ . It is clear from (1) that the set  $W$  is translation congruent modulo  $2\pi$  to  $E$ . Also if  $x \in G$ , then

$$f(\tau_{|G_1}(x)) = h_2(\delta_{|G_1}(x)) = \tau_{|G_2}(\delta_{|G_2}^{-1}(\delta_{|G_1}(x))) = \tau_{|G_2}(x) = \tau_{|G_1}(x)$$

since  $\delta_{|G_2}(x) = \delta_{|G_1}(x)$  and  $\tau_{|G_2}(x) = \tau_{|G_1}(x)$ . This shows that  $\tau_{|G_1}(G) \subset E_0$  and hence  $G \subset W$ . To complete the proof we need to check that  $W$  is dilation congruent modulo 2 to  $E$ . This follows from the facts that  $\delta_{|G_1}((\tau_{|G_1})^{-1}(\hat{E})) = h_1(\hat{E})$ ,  $\delta_{|G_2}((\tau_{|G_2})^{-1}(\hat{E}')) = h_2^{-1}(\hat{E}')$ , and the function  $h$  is a bijection from  $E$  to  $\tilde{E}(= E)$ . In fact one can check that  $W$  is regularized wavelet set.  $\blacksquare$

**Corollary 3.** *For any point  $x_0 \in \mathbb{R} \setminus \{0\}$  there exists an  $\varepsilon > 0$  such that the interval  $I_\varepsilon := (x_0 - \varepsilon, x_0 + \varepsilon)$  is a subwavelet set.*

*Proof.* It suffices to consider the case  $x_0 > 0$ . Choose  $0 < \varepsilon < \min\{\pi/4, x_0/16\}$ . We construct two sets  $G_1$  and  $G_2$  containing  $I_\varepsilon$  and satisfying (a) – (d) in Theorem 2. We write  $E_+ = [\pi, 2\pi)$  and  $E_- = [-2\pi, -\pi)$ . Note that since  $\varepsilon < \min\{\pi, x_0/3\}$  the maps  $\tau_{|I_\varepsilon} : I_\varepsilon \rightarrow E$ ,  $\delta_{|I_\varepsilon} : I_\varepsilon \rightarrow E$  are measurable and injective. (Indeed, if  $\tau(x_1) = \tau(x_2)$  with  $x_1, x_2 \in I_\varepsilon$ , then  $x_1 - x_2 = 2k\pi$  for some  $k \in \mathbb{Z}$ , and since  $|x_1 - x_2| < 2\varepsilon < 2\pi$ , it follows that  $k = 0$  and so  $x_1 = x_2$ . If  $\delta(x_1) = \delta(x_2)$  with  $x_1, x_2 \in I_\varepsilon$ , then  $x_1/x_2 = 2^k$  for some  $k \in \mathbb{Z}$ . Since  $\varepsilon < x_0/3$  we have

$$1/2 < (x_0 - \varepsilon)/(x_0 + \varepsilon) < x_1/x_2 < (x_0 + \varepsilon)/(x_0 - \varepsilon) < 2,$$

and so  $k = 0$  and  $x_1 = x_2$ .) Next we show that since  $\varepsilon < x_0/16$  the set  $E_+ \setminus \delta(I_\varepsilon)$  contains an interval of length greater than  $3\pi/8$ . To see that this is true, we observe that the set

$\delta(I_\varepsilon)$  is either an interval of length  $2^k(2\varepsilon)$ , where the integer  $k$  is uniquely determined by the inequalities  $\pi \leq 2^k x_0 < 2\pi$ , or it is a union of two intervals of combined lengths no more than  $2^{k+1}(2\varepsilon)$ . In the first case, the set  $E_+ \setminus \delta(I_\varepsilon)$  is either an interval or the union of two intervals, and if we assume that each such interval has length no greater than  $3\pi/8$ , we get the following contradiction:

$$\begin{aligned} \pi &= \mu(E_+) = \mu(E_+ \setminus \delta(I_\varepsilon)) + \mu(\delta(I_\varepsilon)) \leq \\ &2(3\pi/8) + 2^k(2\varepsilon) < 3\pi/4 + 2^k(2x_0/16) < \pi. \end{aligned}$$

In the second case (i.e.,  $\delta(I_\varepsilon)$  is a union of intervals), the set  $E_+ \setminus \delta(I_\varepsilon)$  is an interval, and if we assume it has length no larger than  $3\pi/8$ , we get a similar contradiction:

$$\begin{aligned} \pi &= \mu(E_+) = \mu(E_+ \setminus \delta(I_\varepsilon)) + \mu(\delta(I_\varepsilon)) \leq \\ &(3\pi/8) + 2^{k+1}(2\varepsilon) < 3\pi/8 + 2^{k+1}(2x_0/16) < \pi. \end{aligned}$$

Thus  $2^3(E_+ \setminus \delta(I_\varepsilon))$  contains an interval of length greater than  $3\pi$ . Hence there exists  $\ell$  in  $\mathbb{N}$  such that  $E_+ + 2\ell\pi \subset 2^3(E_+ \setminus \delta(I_\varepsilon))$ . We define  $G_1 = (E_- \setminus \tau(I_\varepsilon)) \cup I_\varepsilon \cup ((E_+ \setminus \tau(I_\varepsilon)) + 2\ell\pi)$ . It is clear that  $\tau(G_1) = E$ . Since the maps  $\tau|_{(E_- \setminus \tau(I_\varepsilon))}$ ,  $\tau|_{I_\varepsilon}$ , and  $\tau|_{(E_+ \setminus \tau(I_\varepsilon))}$  are all injective and the sets  $\tau(E_- \setminus \tau(I_\varepsilon))$ ,  $\tau(I_\varepsilon)$ , and  $\tau(E_+ \setminus \tau(I_\varepsilon))$  are pairwise disjoint, it follows that  $\tau|_{G_1}$  is injective and hence is a measurable bijective map. From the choice of  $\ell$  we conclude that  $\delta((E_+ \setminus \tau(I_\varepsilon)) + 2\ell\pi) \subset E_+ \setminus \tau(I_\varepsilon)$ . Hence the sets  $\delta(E_- \setminus \tau(I_\varepsilon))$ ,  $\delta(I_\varepsilon)$ , and  $\delta(E_+ \setminus \tau(I_\varepsilon))$  are pairwise disjoint, and since the maps  $\delta|_{(E_- \setminus \tau(I_\varepsilon))}$ ,  $\delta|_{I_\varepsilon}$ , and  $\delta|_{(E_+ \setminus \tau(I_\varepsilon))}$  are injective, it follows that  $\delta|_{G_1}$  is a measurable injective map. Thus  $G_1$  has the desired properties.

To construct  $G_2$ , we observe first that the collection  $\{2^{-n}E_- + 2\pi\}_{n \in \mathbb{N}} \cup \{2^{-n}E_+ - 2\pi\}_{n \in \mathbb{N}}$  is an interval partition of the set  $E \setminus \{-2\pi\}$ . Moreover  $\tau(I_\varepsilon)$  is either an interval of length  $2\varepsilon$  or the union of two intervals of combined lengths  $2\varepsilon$ . Since  $\varepsilon < \pi/4$ , there exists an  $n_0 \in \mathbb{N}$  such that  $\tau(2^{-n_0}E) \cap \tau(I_\varepsilon) = \emptyset$ . In other words,  $\tau(2^{-n_0}E) \subset E \setminus \tau(I_\varepsilon)$ . We define  $G_2 = I_\varepsilon \cup 2^{-n_0}(E \setminus \delta(I_\varepsilon))$ . Using arguments similar to those above, one shows that  $\delta|_{G_2} : G_2 \rightarrow E$  is a measurable bijective map, and using the fact that

$$\tau(2^{-n_0}(E \setminus \delta(I_\varepsilon))) = (2^{-n_0}(E_- \setminus \delta(I_\varepsilon)) + 2\pi) \cup (2^{-n_0}(E_+ \setminus \delta(I_\varepsilon)) - 2\pi) \subset \tau(2^{-n_0}E) \subset E \setminus \tau(I_\varepsilon),$$

we obtain that  $\tau|_{G_2} : G_2 \rightarrow E$  is a measurable injective map. Thus  $G_2$  has the desired properties, and the proof is complete.  $\blacksquare$

A regularized wavelet set  $W$  is called a *regularized MRA-wavelet set* [2] if the family  $\{\widetilde{W} + 2k\pi\}_{k \in \mathbb{Z}}$  is a partition of  $\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}$ , where  $\widetilde{W} = \bigcup_{n \in \mathbb{N}} 2^{-n}(W)$ . A set is called an *MRA-subwavelet set* if it is a subset of a regularized MRA-wavelet set.

**Question 4.** *Is there a characterization of MRA-subwavelet sets similar to that given in Theorem 2 for subwavelet sets?*

## References

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