2003 LOWER MICHIGAN MATHEMATICS COMPETITION

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Problem 1. Calculate the equations of all lines tangent to both curves $y = x^2$ and $x = y^2 + 1/2$.

Problem 2. Determine all the values of the parameter m such that the equation $\dfrac{2x+1}{x-3} = \dfrac{6x-1}{3x-m}$ has no solution in x.

Problem 3. Let $\triangle ABC$ be an equilateral triangle of side lengths 1. Choose an arbitrary point on $\overline{AB}$, say M, and also select at random a point N on $\overline{AC}$. Calculate the probability that the area of the triangle $\triangle AMN$ is greater than $\dfrac{\sqrt{3}}{8}$.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x + \cos \sqrt{3}x$.
(a) Show that f is not periodic (i.e. there exists no positive number T such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$).
(b) Find a positive number T such that $|f(x + T) - f(x)| \le \frac{1}{100}$ for all $x \in \mathbb{R}$.

Problem 5. Let $\square ABCD$ be a trapezoid with bases $AB = a$ and $CD = b$. Let M be on $\overline{BC}$ and N on $\overline{AD}$ such that the segment $\overline{MN}$ is parallel to the two bases and divides the area of the trapezoid into two equal parts. Find the length of $\overline{MN}$ in terms of a and b.

Problem 6. Let S be a set of positive integers such that for every $x, y \in S$ it follows that $x+y \in S$. Show that there exists an $n \in \mathbb{N}$ and a $d \in \mathbb{N}$ such that if $s \ge n$ then $s \in S$ if and only if s is a multiple of d.

Problem 7. Show that for every positive integer n we have

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor,$$

where $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

Problem 8. Find at least three values of the positive integer M such that $4n + 1$ divides $3n + M$ for exactly two values of $n \in \mathbb{N}$.

Problem 9. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $\int_a^b f(x)dx = 0$. Prove that there exists a point $c \in (a, b)$ such that $f(c) = (f(c) - 1)\int_a^c f(x)dx$.

Problem 10. Count all the numbers divisible by 11 which are obtained by permuting the digits of the number $N = 123456789$.

Problem 11. (Reserved for any contingency) Find an example (with proof) of the following scenario: A and B are 2×2 matrices with integer entries such that $A^2 = I$ and $B^3 = I$ but $(AB)^n \neq I$ for every $n \in \mathbb{N}$, where I denotes the 2×2 identity matrix.